

# National height datum, the Gauss–Listing geoid level value $w_0$ and its time variation $\dot{w}_0$ (Baltic Sea Level Project: epochs 1990.8, 1993.8, 1997.4)

A. Ardalan<sup>1</sup>, E. Grafarend<sup>2</sup>, J. Kakkuri<sup>3</sup>

<sup>1</sup> Department of Surveying and Geomatics Engineering, University of Tehran, P.O. Box 11365-4563, Tehran, Iran  
e-mail: ardalan@ut.ac.ir; Tel.: +98-21-8008841; Fax: +98-21-8008837

<sup>2</sup> Department of Geodesy and GeoInformatics, Stuttgart University, Geschwister-Scholl-Str. 24, D-70174 Stuttgart, Germany  
e-mail: grafarend@gis.uni-stuttgart.de; Tel.: +49-711-1213474; Fax: +49-711-1213285

<sup>3</sup> Finnish Geodetic Institute, Geodeetinrinne 2, 02430 Masala, Finland  
e-mail: juhani.kakkuri@fgi.fi; Tel.: +358-9-295 55 301; Fax: +358-9-295 55 200

Received: 14 August 2000 / Accepted: 19 June 2001

**Abstract.** A methodology for precise determination of the fundamental geodetic parameter  $w_0$ , the potential value of the Gauss–Listing geoid, as well as its time derivative  $\dot{w}_0$ , is presented. The method is based on: (1) ellipsoidal harmonic expansion of the external gravitational field of the Earth to degree/order 360/360 (130 321 coefficients; <http://www.uni-stuttgart.de/gi/research/index.html> projects) with respect to the International Reference Ellipsoid WGD2000, at the GPS positioned stations; and (2) ellipsoidal free-air gravity reduction of degree/order 360/360, based on orthometric heights of the GPS-positioned stations. The method has been numerically tested for the data of three GPS campaigns of the Baltic Sea Level project (epochs 1990.8, 1993.4 and 1997.4). New  $w_0$  and  $\dot{w}_0$  values ( $w_0 = 62\,636\,855.75 \pm 0.21 \text{ m}^2/\text{s}^2$ ,  $\dot{w}_0 = -0.0099 \pm 0.00079 \text{ m}^2/\text{s}^2$  per year,  $w_0/\bar{\gamma} = 6\,379\,781.502 \text{ m}$ ,  $\dot{w}_0/\bar{\gamma} = 1.0 \text{ mm/year}$ , and  $\bar{\gamma} = -9.81802523 \text{ m}^2/\text{s}^2$ ) for the test region (Baltic Sea) were obtained. As by-products of the main study, the following were also determined: (1) the high-resolution sea surface topography map for the Baltic Sea; (2) the most accurate regional geoid amongst four different regional Gauss–Listing geoids currently proposed for the Baltic Sea; and (3) the difference between the national height datums of countries around the Baltic Sea.

**Key words:** Geoid Potential Value – Time Variations – Sea Surface Topography – National Height Datums

## 1 Introduction

There are four quantities, namely  $\{w_0, gm, j_2, \omega\}$ , which are currently accepted by the geodetic community, as the

fundamental geodetic parameters, as documented by, for example Groten (2000). Among these parameters,  $w_0$  plays a crucial role in the geoid determination as well as the computation of the best-fitting reference equipotential surface to the geoid. For example, the reference potential field  $W(r) = gm/r = w_0$  gauged to the geoid potential value  $w_0$  leads to the sphere  $\mathbb{S}_{R=gm/w_0}^2$  as the best-fitting reference equipotential surface to the geoid, or the Somigliana–Pizzetti potential field gauged to the geoid potential value  $w_0$  generates the ellipsoid of revolution  $\mathbb{E}_{a,a,b}^2$ , i.e. WGD2000 (Grafarend and Ardalan 1999), as the best-fitting reference equipotential surface to the geoid. According to Grafarend and Ardalan (1999, p. 614), the accuracy of such a best-fitting ellipsoid is mainly driven by the accuracy of  $w_0$ .

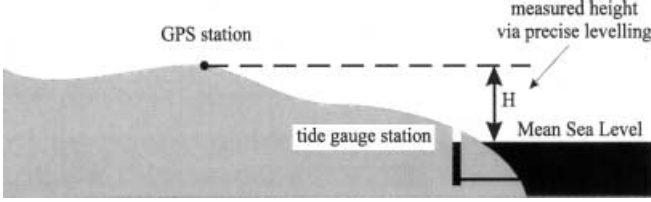
Here we propose the following algorithm for the precise determination of the fundamental geodetic parameter  $w_0$ .

1. Determine the gravity potential at the accurately positioned GPS stations by using the ellipsoidal harmonic expansion of degree/order 360/360 (130 321 coefficients; <http://www.uni-stuttgart.de/gi/research/index.html> projects), plus the ellipsoidal centrifugal potential.
2. Reduce the derived potential values to the mean sea level (MSL) via the precise orthometric heights of GPS-positioned stations, and the ellipsoidal free-air gravity reduction of degree/order 360/360.

The input data for the  $w_0$  operational procedure quoted above can be obtained by GPS observations at permanent tide gauges or at stations which are located close to tide gauge stations and are well connected to the tide gauge stations by precise levelling. The GPS observations collected for Baltic Sea Level project [see, for example, Kakkuri (1990, 1995) and Poutanen and Kakkuri (1999)] provide us with such a source of

information. Figure 1 presents a typical configuration of the GPS and permanent tide gauge stations of Baltic Sea Level Project.

Intuitively speaking, the global geopotential model together with position information derived by GPS provide us with the global gravity information, and, in contrast, the connection of GPS stations to the tide gauge by precise levelling supplies us with the MSL information. Recalling that the geoid according to the Gauss–Listing definition (Listing 1873, p. 45; Gauss 1828, p. 49) is an equipotential surface which fits to the MSL in an optimal way, the approach outlined above should be adequate for a global evaluation of  $w_0$ . However, since the tide gauge observations, aside from random errors, are always affected by some local effects, we may obtain different  $w_0$  values for different tide gauge stations. However, the average value of  $w_{0i}$  data (the index  $i$  runs from one to the total number of tide gauge stations) computed from a set of globally well distributed tide gauges may be considered an unbiased



**Fig. 1.** GPS station, permanent tide gauge station, and measured precise or thometric height of the GPS station above MSL

**Table 3.** Taylor series expansion of the geoid potential value  $w_0$  around the potential value  $W(\lambda, \phi, u)$  at the point  $p\{\lambda, \phi, u\}$

Taylor series expansion of the geoid potential value  $w_0$

$$w_0 = W(\lambda, \phi, u) + \frac{1}{1!} D_u W(\lambda, \phi, u) \cdot (u_0 - u) + \frac{1}{2!} D_u (D_u W(\lambda, \phi, u)) (u_0 - u)^2 + \mathcal{O}((u_0 - u)^3) \quad (6)$$

Partial derivative of potential value  $W$  along the coordinate line of  $u$

$$D_u W = \frac{\partial}{\partial u} W = \sqrt{g_{uu}} \nabla_{\mathbf{e}_u} W \quad (7)$$

Directional derivative of potential value  $W$  along the coordinate line of  $u$

$$\begin{aligned} \nabla_{\mathbf{e}_u} W &:= \langle \text{grad } W(\lambda, \phi, u) \mid \mathbf{e}_u \rangle \\ &= \left\langle \mathbf{e}_\lambda \frac{1}{\sqrt{g_{\lambda\lambda}}} \frac{\partial W}{\partial \lambda} + \mathbf{e}_\phi \frac{1}{\sqrt{g_{\phi\phi}}} \frac{\partial W}{\partial \phi} + \mathbf{e}_u \frac{1}{\sqrt{g_{uu}}} \frac{\partial W}{\partial u} \mid \mathbf{e}_u \right\rangle \\ &= \frac{1}{\sqrt{g_{uu}}} \frac{\partial W(\lambda, \phi, u)}{\partial u} \end{aligned} \quad (8)$$

Taylor series expansion of the geoid potential value  $w_0$  in terms of directional derivative operator

$$\begin{aligned} w_0 &= W(\lambda, \phi, u) + \frac{1}{1!} \nabla_{\mathbf{e}_u} W(\lambda, \phi, u) \cdot \sqrt{g_{uu}} (u_0 - u) \\ &= W(\lambda, \phi, u) + \nabla_{\mathbf{e}_u} (U(\lambda, \phi, \eta) + V(\lambda, \phi, \eta)) \cdot \sqrt{g_{uu}} (u_0 - u) \\ &= W(\lambda, \phi, u) + \frac{1}{\sqrt{g_{uu}}} \frac{\partial (U(\lambda, \phi, u) + V(\lambda, \phi, u))}{\partial u} \cdot \sqrt{g_{uu}} (u_0 - u) \\ &= W(\lambda, \phi, u) + \frac{1}{\sqrt{g_{uu}}} \left( \frac{\partial U(\lambda, \phi, u)}{\partial u} + \frac{\partial V(\lambda, \phi, u)}{\partial u} \right) \cdot \Delta u^{(1)} \\ &= W(\lambda, \phi, u) + \Delta W(\lambda, \phi, u) \end{aligned} \quad (9)$$

subject to

$$\frac{\partial U}{\partial u} = \sum_{n=0}^{360} \sum_{m=-n}^{+n} u_{nm} \frac{1}{Q_{n|m|}^* \left( \frac{b}{e} \right)} \frac{\partial Q_{n|m|}^* \left( \frac{u}{e} \right)}{\partial u} e_{nm}(\lambda, \phi) \quad (10)$$

$$\frac{\partial V}{\partial u} = \omega^2 u \sqrt{u^2 + \varepsilon^2} \cos^2 \phi \quad (11)$$

$$\sqrt{g_{uu}} = \frac{\sqrt{u^2 + \varepsilon^2} \sin^2 \phi}{\sqrt{u^2 + \varepsilon^2}} \quad (12)$$

or, at least, a minimum biased estimate of the geoid potential value  $w_0$ .

In Sect. 2, we set up the theoretical background of our  $w_0$  computation methodology. In Sect. 3 we introduce the data that we will use for various numerical computations. Sections 4, 5 and 6 focus on the results of  $w_0$  computation based on GPS data of the Baltic Sea

**Table 1.** Additive decomposition of the gravity potential  $W$  into gravitational potential  $U$  and centrifugal potential  $V$

$$W(\lambda, \phi, u) = U(\lambda, \phi, u) + V(\lambda, \phi, u) \quad (1)$$

where

$$U(\lambda, \phi, u) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \frac{Q_{n|m|}^* \left( \frac{u}{e} \right)}{Q_{n|m|}^* \left( \frac{b}{e} \right)} e_{nm}(\lambda, \phi) u_{nm} \quad (2)$$

$$\begin{aligned} V(\phi, u) &= \frac{1}{2} \omega^2 (x^2 + y^2) = \frac{1}{2} \omega^2 (u^2 + \varepsilon^2) \cos^2 \phi \\ &= \frac{1}{3} \omega^2 (u^2 + \varepsilon^2) \left( e_{00} - \frac{1}{\sqrt{5}} e_{20}(\lambda, \phi) \right) \end{aligned} \quad (3)$$

and

$$e_{nm}(\lambda, \phi) = P_{n|m|}^*(\sin \phi) \begin{cases} \cos m\lambda & \forall m \geq 0 \\ \sin |m|\lambda & \forall m < 0 \end{cases} \quad (4)$$

**Table 2.** High degree/order reference potential field of the external gravity field of the Earth

$$\begin{aligned} w_0 &\doteq W(\lambda_0, \phi_0, u_0) = \sum_{n=0}^{360} \sum_{m=-n}^{+n} \frac{Q_{n|m|}^* \left( \frac{u_0}{e} \right)}{Q_{n|m|}^* \left( \frac{b}{e} \right)} e_{nm}(\lambda_0, \phi_0) u_{nm} \\ &\quad + \frac{1}{3} \omega^2 (u_0^2 + \varepsilon^2) \left( e_{00} - \frac{1}{\sqrt{5}} e_{20}(\lambda_0, \phi_0) \right) \end{aligned} \quad (5)$$

Level Project, first, second and third campaigns. Section 6 leads to the best estimate of  $w_0 = 62\,636\,855.75 \pm 0.21 \text{ m}^2/\text{s}^2$  with a remarkably small mean square error. In Sect. 7 we aim at an estimate of the time derivative of  $w_0$ , namely  $\dot{w}_0 = -0.0099 \pm 0.00079 \text{ m}^2/\text{s}^2$  per year. Section 8 discusses the apparent difference in the height datum between various countries around the Baltic Sea as well as the transfer function of gravity potential differences into geometric heights in order to obtain the differences between height datums. For the unification of national, regional or continental to a *global* height datum we acknowledge the contributions of Rummel and Teunissen (1988); Heck and Rummel (1990); Xu and Rummel (1991); Xu (1992); Rapp (1994); Rummel and Ilk (1995); Sansò and Usai (1995). Finally, Sect. 9 concentrates on the determination of the sea surface topography of the Baltic Sea at the tide gauges of the Baltic Sea Level Project.

#### $W_0/W_{0i}$ computations: problem formulation

It is well known that a slowly uniformly rotating, self-gravitating liquid body of radial mass distribution forms

**Table 5.** Operational procedure for computing the geoid potential value  $w_0$

$$w_0 = \sum_{n=0}^{360} \sum_{m=-n}^{+n} u_{nm} \frac{Q_{n|m|}^*(\frac{u}{\varepsilon})}{Q_{n|m|}^*(\frac{b}{\varepsilon})} e_{nm}(\lambda, \phi) + \frac{1}{2} \omega^2 (u^2 + \varepsilon^2) \cos^2 \phi$$

$$+ \frac{1}{\sqrt{g_{uu}}} \left( \sum_{n=0}^{360} \sum_{m=-n}^{+n} u_{nm} \frac{1}{Q_{n|m|}^*(\frac{b}{\varepsilon})} \frac{\partial Q_{n|m|}^*(\frac{b}{\varepsilon})}{\partial u} e_{nm}(\lambda, \phi) + \omega^2 u \sqrt{u^2 + \varepsilon^2} \cos^2 \phi \right) (-H_0^p)$$

$$= W(\lambda, \phi, u) + \Delta W(\lambda, \phi, u; H_0^p)$$

subject to:

(i) the gravity potential at point  $p$

$$W(\lambda, \phi, u) = \sum_{n=0}^{360} \sum_{m=-n}^{+n} u_{nm} \frac{Q_{n|m|}^*(\frac{u}{\varepsilon})}{Q_{n|m|}^*(\frac{b}{\varepsilon})} e_{nm}(\lambda, \phi) + \frac{1}{2} \omega^2 (u^2 + \varepsilon^2) \cos^2 \phi$$

(ii) ellipsoidal free-air reduction of degree/order 360/360

$$\Delta W(\lambda, \phi, u; H_0^p) = \frac{1}{\sqrt{g_{uu}}} \left( \sum_{n=0}^{360} \sum_{m=-n}^{+n} u_{nm} \frac{1}{Q_{n|m|}^*(\frac{b}{\varepsilon})} \frac{\partial Q_{n|m|}^*(\frac{u}{\varepsilon})}{\partial u} e_{nm}(\lambda, \phi) + \omega^2 u \sqrt{u^2 + \varepsilon^2} \cos^2 \phi \right) (-H_0^p)$$

**Table 4.** Partial derivative of associated Legendre functions of second kind with respect to  $u$

$$\begin{aligned} &\text{Recursive relations} \\ &\frac{\partial Q_{0,0}^*(\frac{u}{\varepsilon})}{\partial u} = -\frac{a}{\sqrt{u^2 + \varepsilon^2}} \\ &\frac{\partial Q_{n,0}^*(\frac{u}{\varepsilon})}{\partial u} = n \frac{u}{\sqrt{u^2 + \varepsilon^2}} Q_{n,0}^*(\frac{u}{\varepsilon}) - (2n+1) \frac{a}{\sqrt{u^2 + \varepsilon^2}} Q_{n-1,0}^*(\frac{u}{\varepsilon}) \\ &\frac{\partial Q_{n,m}^*(\frac{u}{\varepsilon})}{\partial u} = -(n-m+1) Q_{n,m-1}^*(\frac{u}{\varepsilon}) - m \frac{u}{\sqrt{u^2 + \varepsilon^2}} Q_{n,m}^*(\frac{u}{\varepsilon}) \quad \forall m \in [1, n] \end{aligned}$$

**Table 6.** Cartesian coordinates of GPS stations of the Baltic Sea Level project, first campaign, 1990.8, in ITRF 91 reference frame

| Station name <sup>a</sup> | X (m)          | Y (m)          | Z (m)          |
|---------------------------|----------------|----------------|----------------|
| Borkum (Ger)              | 3 770 668.4100 | 446 076.5560   | 5 107 686.4450 |
| Degerby (Fin)             | 2 994 005.5910 | 1 112 565.2290 | 5 502 270.8220 |
| Furuögrund (Swe)          | 2 527 022.8530 | 981 957.3370   | 5 753 940.7500 |
| Hamina (Fin)              | 2 795 471.3860 | 1 435 427.7410 | 5 531 682.1280 |
| Hanko (Fin)               | 2 959 173.1020 | 1 254 706.6550 | 5 490 604.5210 |
| Helgoland (Ger)           | 3 706 045.0350 | 513 713.1770   | 5 148 193.1900 |
| Helsinki (Fin)            | 2 885 134.8290 | 1 342 693.6440 | 5 509 043.9240 |
| Kemi (Fin)                | 2 397 170.2880 | 1 093 246.9200 | 5 789 077.1960 |
| Klagshamn (Swe)           | 3 527 585.8260 | 807 513.8050   | 5 234 549.4560 |
| Klaipeda (Lit)            | —              | —              | —              |
| Kronstadt (Rus)           | —              | —              | —              |
| List/Sylt (Ger)           | 3 625 340.1590 | 5 378 854.8410 | 5 202 539.3040 |
| Mäntyluoto (Fin)          | 2 831 096.8750 | 1 113 102.7630 | 5 587 164.9240 |
| Molas (Lit)               | —              | —              | —              |
| Ölands N. U. (Swe)        | 3 295 551.6790 | 1 012 564.8840 | 5 348 113.5190 |
| Raahe (Fin)               | 2 492 699.6820 | 1 131 503.6280 | 5 741 504.1800 |
| Ratan (Swe)               | 2 620 087.7400 | 1 000 008.3720 | 5 709 322.5000 |
| Shepelevo (Rus)           | —              | —              | —              |
| Spikarna (Swe)            | 2 828 573.5470 | 893 623.7640   | 5 627 446.8610 |
| Stockholm (Swe)           | 3 101 011.4900 | 1 013 009.1090 | 5 462 375.0730 |
| Swinoujscie (Pol)         | 3 649 458.4740 | 927 709.8850   | 5 130 741.4690 |
| Ustka (Pol)               | 3 545 014.4560 | 1 073 939.6850 | 5 174 949.7730 |
| Vaasa (Fin)               | 2 691 307.2790 | 1 063 691.5640 | 5 664 806.2010 |
| Visby (Swe)               | 3 249 304.5320 | 1 073 624.8010 | 5 364 362.8610 |
| Warnemünde (Ger)          | 3 658 230.7910 | 783 507.3140   | 5 148 395.8730 |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

in its equilibrium state an ellipsoid of revolution at its boundary. Therefore, the Earth as a massive body, which once was a liquid at hydrostatic equilibrium, globally does not deviate considerably from an oblate spheroid or ellipsoid of revolution. Therefore, the

external gravity field of the Earth can most conveniently be presented in terms of an ellipsoidal model as explained in Table 1.

According to Table 1, we additively decomposed the gravity potential  $W$  into the gravitational potential  $U$

**Table 7.** Cartesian coordinates of GPS stations of the Baltic Sea Level project, second campaign, 1993.4, in ITRF 93 reference frame

| Station name <sup>a</sup> | $X$ (m)        | $Y$ (m)        | $Z$ (m)        |
|---------------------------|----------------|----------------|----------------|
| Borkum (Ger)              | 3 770 668.1000 | 446 076.4240   | 5 107 686.2360 |
| Degerby (Fin)             | 2 994 005.4960 | 1 112 565.2460 | 5 502 270.9240 |
| Furuögrun (Swe)           | 2 527 022.9170 | 981 957.2520   | 5 753 940.9600 |
| Hamina (Fin)              | 2 795 471.2440 | 1 435 427.7570 | 5 531 682.2020 |
| Hanko (Fin)               | 2 959 172.9990 | 1 254 706.6680 | 5 490 604.6630 |
| Helgoland (Ger)           | 3 706 044.9580 | 513 713.1520   | 5 148 193.3740 |
| Helsinki (Fin)            | 2 885 134.7810 | 1 342 693.6430 | 5 509 044.0630 |
| Kemi (Fin)                | 2 397 170.2290 | 1 093 246.8740 | 5 789 077.1460 |
| Klagshamn (Swe)           | 3 527 585.8220 | 807 513.8260   | 5 234 549.6830 |
| Klaipeda (Lit)            | 3 353 590.2730 | 1 302 062.9830 | 5 249 159.3680 |
| Kronstadt (Rus)           | —              | —              | —              |
| List/Sylt (Ger)           | 3 625 340.0140 | 537 853.8050   | 5 202 539.0420 |
| Mäntyluoto (Fin)          | 2 831 096.7560 | 1 113 102.7150 | 5 587 165.0170 |
| Molas (Lit)               | 3 358 793.4290 | 1 294 907.3540 | 5 247 584.3520 |
| Ölands N. U. (Swe)        | 3 295 551.6450 | 1 012 564.8500 | 5 348 113.6740 |
| Raahe (Fin)               | 2 493 889.7970 | 1 131 220.2470 | 5 741 045.9690 |
| Ratan (Swe)               | 2 620 087.6710 | 1 000 008.2270 | 5 709 322.5780 |
| Shepelevo (Rus)           | 2 796 394.3740 | 1 556 360.1010 | 5 498 639.2920 |
| Spikarna (Swe)            | 2 828 573.5140 | 893 623.6890   | 5 627 447.0490 |
| Stockholm (Swe)           | 3 101 011.4990 | 1 013 009.1210 | 5 462 375.2820 |
| Swinoujscie (Pol)         | 3 649 458.4600 | 927 709.9170   | 5 130 741.6590 |
| Ustka (Pol)               | 3 545 014.3980 | 1 073 939.7300 | 5 174 949.9340 |
| Vaasa (Fin)               | 2 691 307.2900 | 1 063 691.4760 | 5 664 806.3350 |
| Visby (Swe)               | 3 249 304.5220 | 1 073 624.8500 | 5 364 363.1060 |
| Warnemünde (Ger)          | 3 658 217.6960 | 783 004.6440   | 5 148 504.2800 |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 8.** Cartesian coordinates of GPS stations of the Baltic Sea Level project, third campaign project, 1997.4, in ITRF 96 reference frame

| Station name <sup>a</sup> | $X$ (m)        | $Y$ (m)        | $Z$ (m)        |
|---------------------------|----------------|----------------|----------------|
| Borkum (Ger)              | 3 770 667.9990 | 446 076.4900   | 5 107 686.2080 |
| Degerby (Fin)             | 2 994 064.9360 | 1 112 559.0570 | 5 502 241.3760 |
| Furuögrund (Swe)          | 2 527 022.8720 | 981 957.2890   | 5 753 940.9920 |
| Hamina (Fin)              | 2 795 471.2050 | 1 435 427.7920 | 5 531 682.2000 |
| Hanko (Fin)               | 2 959 210.9710 | 1 254 679.1200 | 5 490 594.4410 |
| Helgoland (Ger)           | 3 706 044.9440 | 513 713.2150   | 5 148 193.4470 |
| Helsinki (Fin)            | 2 885 137.3910 | 1 342 710.2300 | 5 509 039.1190 |
| Kemi (Fin)                | 2 397 071.5770 | 1 093 330.3130 | 5 789 108.4470 |
| Klagshamn (Swe)           | 3 527 585.7670 | 807 513.8950   | 5 234 549.7020 |
| Klaipeda (Lit)            | 3 353 590.2430 | 1 302 063.0140 | 5 249 159.4120 |
| Kronstadt (Rus)           | 2 776 311.8190 | 1 587 590.1310 | 5 499 880.1330 |
| List/Sylt (Ger)           | 3 625 339.9122 | 537 853.8700   | 5 202 539.0260 |
| Mäntyluoto (Fin)          | 2 831 096.7190 | 1 113 102.7640 | 5 587 165.0460 |
| Molas (Lit)               | 3 358 793.3810 | 1 294 907.4050 | 5 247 584.4010 |
| Ölands N. U. (Swe)        | 3 295 551.5710 | 1 012 564.9060 | 5 348 113.6690 |
| Raahe (Fin)               | 2 494 035.0240 | 1 131 370.9940 | 5 740 955.4100 |
| Ratan (Swe)               | 2 620 087.6290 | 1 000 008.2700 | 5 709 322.6040 |
| Shepelevo (Rus)           | 2 796 394.9140 | 1 556 363.7830 | 5 498 638.0600 |
| Spikarna (Swe)            | 2 828 573.4640 | 893 623.7290   | 5 627 447.0690 |
| Stockholm (Swe)           | 3 101 008.8620 | 1 013 021.0370 | 5 462 373.3830 |
| Swinoujscie (Pol)         | 3 648 326.5170 | 924 984.0310   | 5 132 035.2720 |
| Ustka (Pol)               | 3 545 014.3300 | 1 073 939.7720 | 5 174 949.9470 |
| Vaasa (Fin)               | 2 691 307.2540 | 1 063 691.5240 | 5 664 806.3800 |
| Visby (Swe)               | 3 249 304.4370 | 1 073 624.8910 | 5 364 363.0730 |
| Warnemünde (Ger)          | 3 658 231.7070 | 783 518.3220   | 5 148 404.3509 |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

and the centrifugal potential  $V$ , both represented in terms of Jacobi ellipsoidal coordinates  $\{\lambda, \phi, u\}$ , also called spheroidal. In the space external to the Earth, its gravitational potential is harmonic, namely it is an ele-

ment of the three-dimensional (3-D) Laplace equation with associated Legendre functions of the first  $P_{nm}^*(\sin \phi)$  and of the second kind  $Q_{nm}^*(u/b)$  as well as  $\{\cos m\lambda, \sin |m|\lambda\}$  as eigenfunctions. Accordingly, in

**Table 9.** Jacobi ellipsoidal coordinates of the GPS stations of the Baltic Sea Level project, first campaign, 1990.8, with respect to the reference ellipsoid of WGD2000 in the mean tide system,  $a = 6\,378\,136.701 \pm 0.053$  m and  $b = 6\,356\,751.661 \pm 0.052$  m

| Station name <sup>a</sup> | $\lambda$ |    |        | $\phi$ |    |        | $u$ (m)        |
|---------------------------|-----------|----|--------|--------|----|--------|----------------|
|                           | °         | '  | ''     | °      | '  | ''     |                |
| Borkum (Ger)              | 6         | 44 | 48.592 | 53     | 27 | 56.290 | 6 356 797.4401 |
| Degerby (Fin)             | 20        | 23 | 5.799  | 59     | 56 | 54.812 | 6 356 772.7694 |
| Furuögrund (Swe)          | 21        | 14 | 6.953  | 64     | 50 | 43.972 | 6 356 784.9708 |
| Hamina (Fin)              | 27        | 10 | 47.061 | 60     | 28 | 56.235 | 6 356 769.0382 |
| Hanko (Fin)               | 22        | 58 | 38.020 | 59     | 44 | 21.212 | 6 356 773.7771 |
| Helgoland (Ger)           | 7         | 53 | 30.345 | 54     | 5  | 0.468  | 6 356 795.7627 |
| Helsinki (Fin)            | 24        | 57 | 23.340 | 60     | 4  | 14.304 | 6 356 776.0350 |
| Kemi (Fin)                | 24        | 30 | 56.527 | 65     | 36 | 5.357  | 6 356 772.7375 |
| Klagshamn (Swe)           | 12        | 53 | 37.154 | 55     | 25 | 56.873 | 6 356 790.0414 |
| Klaipeda (Lit)            | —         | —  | —      | —      | —  | —      | —              |
| Kronstadt (Rus)           | —         | —  | —      | —      | —  | —      | —              |
| List/Sylt (Ger)           | 56        | 1  | 12.319 | 38     | 47 | 11.504 | 8 305 186.1308 |
| Mäntyluoto (Fin)          | 21        | 27 | 47.774 | 61     | 30 | 49.223 | 6 356 773.4710 |
| Molas (Lit)               | —         | —  | —      | —      | —  | —      | —              |
| Ölands N. U. (Swe)        | 17        | 4  | 46.851 | 57     | 16 | 48.643 | 6 356 783.6250 |
| Raahe (Fin)               | 24        | 24 | 52.486 | 64     | 35 | 0.055  | 6 356 772.4568 |
| Ratan (Swe)               | 20        | 53 | 25.243 | 63     | 54 | 56.307 | 6 356 775.1165 |
| Shepelevo (Rus)           | —         | —  | —      | —      | —  | —      | —              |
| Spikarna (Swe)            | 17        | 31 | 57.907 | 62     | 17 | 3.822  | 6 356 779.3644 |
| Stockholm (Swe)           | 18        | 5  | 26.484 | 59     | 14 | 16.253 | 6 356 788.2619 |
| Swinoujscie (Pol)         | 14        | 15 | 45.951 | 53     | 48 | 58.479 | 6 356 790.0989 |
| Ustka (Pol)               | 16        | 51 | 13.868 | 54     | 29 | 48.357 | 6 356 786.0870 |
| Vaasa (Fin)               | 21        | 33 | 55.917 | 63     | 1  | 3.025  | 6 356 771.3215 |
| Visby (Swe)               | 18        | 17 | 3.922  | 57     | 33 | 7.925  | 6 356 779.3580 |
| Warnemünde (Ger)          | 12        | 5  | 19.600 | 54     | 5  | 11.799 | 6 356 793.1157 |

<sup>a</sup>Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 10.** Jacobi ellipsoidal coordinates of the GPS stations of the Baltic Sea Level project, second campaign, 1993.4, with respect to the reference ellipsoid of WGD2000 in the mean tide system,  $a = 6\,378\,136.701 \pm 0.053$  m and  $b = 6\,356\,751.661 \pm 0.052$  m

| Station name <sup>a</sup> | $\lambda$ |    |        | $\phi$ |    |        | $u$ (m)        |
|---------------------------|-----------|----|--------|--------|----|--------|----------------|
|                           | °         | '  | ''     | °      | '  | ''     |                |
| Borkum (Ger)              | 6         | 44 | 48.587 | 53     | 27 | 56.294 | 6 356 797.0766 |
| Degerby (Fin)             | 20        | 23 | 5.802  | 59     | 56 | 54.816 | 6 356 772.8204 |
| Furuögrund (Swe)          | 21        | 14 | 6.945  | 64     | 50 | 43.974 | 6 356 785.1749 |
| Hamina (Fin)              | 27        | 10 | 47.066 | 60     | 28 | 56.239 | 6 356 769.0445 |
| Hanko (Fin)               | 22        | 58 | 38.023 | 59     | 44 | 21.217 | 6 356 773.8537 |
| Helgoland (Ger)           | 7         | 53 | 30.344 | 54     | 5  | 0.474  | 6 356 795.8647 |
| Helsinki (Fin)            | 24        | 57 | 23.341 | 60     | 4  | 14.307 | 6 356 776.1370 |
| Kemi (Fin)                | 24        | 30 | 56.525 | 65     | 36 | 5.358  | 6 356 772.6609 |
| Klagshamn (Swe)           | 12        | 53 | 37.155 | 55     | 25 | 56.877 | 6 356 790.2264 |
| Klaipeda (Lit)            | 21        | 13 | 9.013  | 55     | 39 | 54.147 | 6 356 805.2088 |
| Kronstadt (Rus)           | —         | —  | —      | —      | —  | —      | —              |
| List/Syt (Ger)            | 8         | 26 | 19.755 | 54     | 55 | 37.511 | 6 356 797.0447 |
| Mäntyluoto (Fin)          | 21        | 27 | 47.774 | 61     | 30 | 49.228 | 6 356 773.4901 |
| Molas (Lit)               | 21        | 4  | 58.889 | 55     | 38 | 24.697 | 6 356 781.6159 |
| Ölands N. U. (Swe)        | 17        | 4  | 46.850 | 57     | 16 | 48.646 | 6 356 783.7334 |
| Raahe (Fin)               | 24        | 23 | 55.998 | 64     | 34 | 25.438 | 6 356 772.1890 |
| Ratan (Swe)               | 20        | 53 | 25.235 | 63     | 54 | 56.312 | 6 356 775.1357 |
| Shepelevo (Rus)           | 29        | 5  | 54.760 | 59     | 52 | 59.790 | 6 356 771.8318 |
| Spikarna (Swe)            | 17        | 31 | 57.902 | 62     | 17 | 3.826  | 6 356 779.5047 |
| Stockholm (Swe)           | 18        | 5  | 26.485 | 59     | 14 | 16.256 | 6 356 779.5047 |
| Swinoujscie (Pol)         | 14        | 15 | 45.953 | 53     | 48 | 58.483 | 6 356 788.4469 |
| Ustka (Pol)               | 16        | 51 | 13.872 | 54     | 29 | 48.361 | 6 356 790.2519 |
| Vaasa (Fin)               | 21        | 33 | 55.911 | 63     | 1  | 3.028  | 6 356 786.1954 |
| Visby (Swe)               | 18        | 17 | 3.925  | 57     | 33 | 7.929  | 6 356 779.5685 |
| Warnemünde (Ger)          | 12        | 4  | 52.651 | 54     | 5  | 16.952 | 6 356 811.9505 |

<sup>a</sup>Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

terms of surface spheroidal harmonics  $e_{nm}(\lambda, \phi)$  given by Eq. (4), Eqs. (1)–(3) are proper representations. For the definition of the Jacobi ellipsoidal coordinates  $\{\lambda, \phi, u\}$  we refer to Appendix A. Appendix B provides the definition of the normalised associated Legendre functions of the first as well as of the second kind. Finally, Appendix C outlines how the ellipsoidal harmonic coefficients  $u_{nm}$  can be obtained from the available spherical harmonic coefficients.

Equation (1) is exact and without any approximation. However, in practice we have to approximate it by using a limited number of ellipsoidal harmonic coefficients, say up to degree/order 360/360. Consequently, we arrive at an approximate form of Eq. (1), which can serve as a reference gravity field of the Earth. Such a reference field covers the gravity effects of features which are bigger than 50–60 km.

Roughly speaking, if we now place the GPS receiver at point  $p_0$  on the MSL (i.e. geoid) and measure the Cartesian coordinates  $\{x_0, y_0, z_0\}$  of  $p_0$ , then once we have converted these Cartesian coordinates into Jacobi ellipsoidal coordinates  $\{\lambda_0, \phi_0, u_0\}$  (via the transformation relations given in Appendix A) an estimation of geoid potential value  $w_0$  can be obtained according to Table 2.

Since at the geoid's level the variation of potential is quite smooth, degree/order 360/360 is enough for the computation of the geoid value at a level of accuracy of better than  $0.5 \text{ m}^2/\text{s}^2$ . Note that the high-frequency changes of the gravity field are mainly due to the topographical masses, therefore at the geoid's level we do not have any high-frequency variations and as such the ellipsoidal harmonic expansion of degree/order 360/360 can satisfactorily represent the geoid potential value  $w_0$ .

In practice however, we may place the GPS receiver at point  $p$ , which is in the vicinity of a tide gauge station, and measure the Cartesian coordinates  $\{x, y, z\}$ . These Cartesian coordinates, once converted into Jacobi ellipsoidal coordinates  $\{\lambda, \phi, u\}$ , can provide us with the gravity potential  $W(\lambda, \phi, u)$  at the point  $p$ , which is of course different from the geoid potential value  $w_0$ .

Knowing that the potential difference between the point  $p\{\lambda, \phi, u\}$  and the point  $p_0\{\lambda_0, \phi_0, u_0\}$  on the geoid is only due to the orthometric height of  $p\{\lambda, \phi, u\}$ , the geoid potential value  $w_0$  can be written in terms of the Taylor series expansion around the potential value  $W(\lambda, \phi, u)$  at the point  $p\{\lambda, \phi, u\}$  as explained in Table 3.

In Table 3  $D_u W$  is the partial derivative of potential  $W(\lambda, \phi, u)$  with respect to  $u$ , and  $(u_0 - u)$  is the height of the GPS station above MSL, of course in terms of the Jacobi ellipsoidal height component. The partial derivative  $D_u W$  is related to directional derivative of  $W$  along the coordinate line of  $u$  as introduced by Eq. (7). For the definition of directional derivative  $\nabla_{\mathbf{e}_u} W$  along the coordinate line of  $u$ , we refer to Eq. (8).  $\mathbf{e}_\lambda = \mathbf{t}_\lambda / \sqrt{g_{\lambda\lambda}}$ ,  $\mathbf{e}_\phi = \mathbf{t}_\phi / \sqrt{g_{\phi\phi}}$  and  $\mathbf{e}_u = \mathbf{t}_u / \sqrt{g_{uu}}$  are orthonormal base vectors of the Jacobi ellipsoidal coordinates  $\{\lambda, \phi, u\}$ , and  $\sqrt{g_{\lambda\lambda}}$ ,  $\sqrt{g_{\phi\phi}}$ , and  $\sqrt{g_{uu}}$  are metric tensors. See Appendix A, Eq. (A10) for the definition of the metric tensors in terms of Jacobi ellipsoidal coordinates  $\{\lambda, \phi, u\}$ . The vectors  $\mathbf{t}_\lambda = \partial \mathbf{x} / \partial \lambda$ ,  $\mathbf{t}_\phi = \partial \mathbf{x} / \partial \phi$  and

$\mathbf{t}_u = \partial \mathbf{x} / \partial u$  are tangent to coordinate lines of Jacobi ellipsoidal coordinates  $\lambda, \phi$  and  $u$ , respectively. Therefore, Eq. (6) up to the terms of order of magnitude  $\mathcal{O}(u_0 - u)^2$  can be written as Eq. (9).

According to Eringen (1962, p. 437),  $\Delta u^{(1)} = \sqrt{g_{uu}}(u_0 - u)$  is the physical component of the Jacobi ellipsoidal height difference  $(u_0 - u)$ . The partial derivative of associated Legendre functions of the second kind with respect to  $u$  can be derived from the recursive relations of Table 4.

We call the second term of Eq. (9), i.e.  $\Delta W(\lambda, \phi, u)$ , the ‘ellipsoidal free-air reduction’ of the potential value  $W(\lambda, \phi, u)$  to the geoid's surface. Since in the ellipsoidal harmonic expansion all masses of the Earth are condensed inside the reference ellipsoid of WGD2000, in the eyes of the ellipsoidal harmonic model, we actually have free air between  $p$  and  $p_0$ . The physical component  $\sqrt{g_{uu}}(u_0 - u)$  here can be interpreted as the orthometric height  $H_0$  of the GPS station with opposite sign, i.e.  $\sqrt{g_{uu}}(u_0 - u) = -H_0$ .

Therefore, the geoid potential value  $w_0$  can be derived from the Jacobi coordinates  $\{\lambda, \phi, u\}$  of the GPS station  $p$  and the orthometric height of  $p$ ,  $H_0^p$ , as outlined in Table 5.

Having set up the theoretical foundation of the  $w_0$  computation problem, we can begin our case study by computing  $w_0$  and  $\dot{w}_0$  from the GPS observation of the Baltic Sea Level Project, first, second, and third campaigns, in the following sections.

### 3. $W_0$ computations: input data

Based on the method described in the previous section, here we compute the  $w_0$  value via the GPS observations of three successive GPS campaigns of the Baltic Sea Level Project. For a review of the state of the art of the Baltic Sea Level Project we recommend Kakkuri (1990, 1995); Poutanen and Kakkuri (1999). Tables 6–8 present the Cartesian coordinates of the GPS stations of the Baltic Sea Level Project, first, second and third campaigns, respectively. Table 9–11 present the computed Jacobi ellipsoidal coordinates (see Appendix A for transformation relations) of those stations given in Tables 6–8. The ellipsoidal harmonic coefficients needed for the series expansion of Eq. (15) and Eq. (16) are provided via the transformation of the spherical harmonic coefficients of EGM96 (Lemoine et al. 1998) to ellipsoidal ones with respect to the reference ellipsoid of WGD2000 through the exact transformation relations given in Appendix C. The ellipsoidal harmonic coefficients are computed in a mean-tide/permanent-tide system [see Ekman (1996) for the definition of various permanent-tide system]. For this purpose we first transferred the second zonal spherical harmonic coefficients of the EGM96 geopotential model from a tide-free system into a mean-tide system, and then applied the transformation machinery of Appendix C to obtain ellipsoidal harmonic coefficients from spherical harmonic coefficients. We have made the computed ellipsoidal harmonic coefficient available to the public, and inter-

ested readers can download the ellipsoidal harmonic coefficients, plus the manual for using them, from the

home page of the Geodetic Institute of the Stuttgart University (<http://www.uni-stuttgart.de/gi/research/in->

**Table 11.** Jacobi ellipsoidal coordinates of the GPS stations of the Baltic Sea Level project, third campaign, 1997.4, with respect to the reference ellipsoid of WGD2000 in the mean tide system,  $a = 6378\,136.701 \pm 0.053$  m and  $b = 6356\,751.661 \pm 0.052$  m

| Station name <sup>a</sup> | $\lambda$ |    |        | $\phi$ |    |        | $u$ (m)        |
|---------------------------|-----------|----|--------|--------|----|--------|----------------|
|                           | °         | '  | ''     | °      | '  | ''     |                |
| Borkum (Ger)              | 6         | 44 | 48.591 | 53     | 27 | 56.296 | 6 356 797.0000 |
| Degerby (Fin)             | 20        | 23 | 4.091  | 59     | 56 | 52.837 | 6 356 773.9748 |
| Furuögrund(Swe)           | 21        | 14 | 6.949  | 64     | 50 | 43.975 | 6 356 785.1940 |
| Hamina (Fin)              | 27        | 10 | 47.069 | 60     | 28 | 56.240 | 6 356 769.0318 |
| Hanko (Fin)               | 22        | 58 | 35.444 | 59     | 44 | 20.373 | 6 356 777.1894 |
| Helgoland (Ger)           | 7         | 53 | 30.348 | 54     | 5  | 0.476  | 6 356 795.9221 |
| Helsinki (Fin)            | 24        | 57 | 24.245 | 60     | 4  | 13.965 | 6 356 776.5070 |
| Kemi (Fin)                | 24        | 31 | 5.674  | 65     | 36 | 7.401  | 6 356 778.4778 |
| Klagshamn (Swe)           | 12        | 53 | 37.160 | 55     | 25 | 56.878 | 6 356 790.2200 |
| Klaipeda (Lit)            | 21        | 13 | 9.016  | 55     | 39 | 54.158 | 6 356 805.2343 |
| Kronstadt (Rus)           | 29        | 45 | 44.638 | 59     | 54 | 20.012 | 6 356 772.6865 |
| List/Sylt (Ger)           | 8         | 26 | 19.759 | 54     | 55 | 37.513 | 6 356 796.9809 |
| Mäntyluoto (Fin)          | 21        | 27 | 47.778 | 61     | 30 | 49.229 | 6 356 773.5092 |
| Molas (Lit)               | 21        | 4  | 58.893 | 55     | 38 | 24.698 | 6 356 781.6414 |
| Ölands N. U. (Swe)        | 17        | 4  | 46.854 | 57     | 16 | 48.648 | 6 356 783.6952 |
| Raahe (Fin)               | 24        | 24 | 1.820  | 64     | 34 | 18.495 | 6 356 773.6451 |
| Ratan (Swe)               | 20        | 53 | 25.239 | 63     | 54 | 56.313 | 1 022 53.32707 |
| Shepelevo (Rus)           | 29        | 5  | 54.951 | 59     | 52 | 59.707 | 6 356 771.9019 |
| Spikarna (Swe)            | 17        | 31 | 57.906 | 62     | 17 | 3.827  | 6 356 779.5047 |
| Stockholm (Swe)           | 18        | 5  | 27.253 | 59     | 14 | 16.192 | 6 356 787.4200 |
| Swinoujscie (Pol)         | 14        | 13 | 36.455 | 53     | 50 | 9.408  | 6 356 794.5253 |
| Ustka (Pol)               | 16        | 51 | 13.875 | 54     | 29 | 48.363 | 6 356 786.1763 |
| Vaasa (Fin)               | 21        | 33 | 55.915 | 63     | 1  | 3.029  | 6 356 771.4619 |
| Visby (Swe)               | 18        | 17 | 3.929  | 57     | 33 | 7.931  | 6 356 779.5047 |
| Warnemünde (Ger)          | 12        | 5  | 20.183 | 54     | 5  | 11.875 | 6 356 801.8730 |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 12.** Orthometric height of the GPS stations of the Baltic Sea Level project, first campaign, 1990.8, derived from different sources (cases 1–5)

| Station name <sup>a</sup> | Orthometric height<br>(case 1) (Poutanen et al. 1999) | Orthometric height<br>(case 2) (Kakkuri 1995) | Orthometric height<br>(case 3) (Kakkuri and Poutanen 1997) | Orthometric height<br>(case 4) (Poutanen et al. 1999) | Orthometric height<br>(case 5) (EGG97) |
|---------------------------|---|---|--|---|--|
| Borkum (Ger)              | 4.581   | 5.454   | —  | 5.884   | 5.556                                  |
| Degerby (Fin)             | 1.695   | 2.345   | 2.428  | 2.541   | 2.422                                  |
| Furuögrund (Swe)          | 10.912  | 11.597  | 11.541   | 11.830  | 12.180                                 |
| Hamina (Fin)              | 1.619   | 2.374   | 2.357  | 2.645   | 2.458                                  |
| Hanko (Fin)               | 1.762   | 2.434   | 2.473  | 2.694   | 2.626                                  |
| Helgoland (Ger)           | 4.539   | —   | —  | 5.196   | 4.997                                  |
| Helsinki (Fin)            | 6.033   | 6.654   | 6.663  | 6.933   | 6.820                                  |
| Kemi (Fin)                | 1.246   | 2.161   | 2.248  | 2.546   | 2.485                                  |
| Klagshamn (Swe)           | 2.039   | 2.551   | 2.169  | 2.775   | 2.759                                  |
| Klaipeda (Lit)            | —   | —   | —  | —   | —                                      |
| Kronstadt (Rus)           | —   | —   | —  | —   | —                                      |
| List/Sylt (Ger)           | 4.160   | 4.904   | —  | 5.302   | 5.135                                  |
| Mäntyluoto (Fin)          | 2.467   | 3.208   | 3.222  | 3.451   | —                                      |
| Molas (Lit)               | —   | —   | —  | —   | —                                      |
| Ölands N. U. (Swe)        | 4.118   | 4.493   | 4.714  | 4.961   | 4.729                                  |
| Raahe (Fin)               | 2.289   | 3.127   | 3.168  | 3.510   | 3.411                                  |
| Ratan (Swe)               | 1.476   | 2.361   | 2.268  | 2.581   | 2.633                                  |
| Shepelevo (Rus)           | —   | —   | —  | —   | —                                      |
| Spikarna (Swe)            | 1.872   | 2.938   | 2.603  | 2.809   | 2.943                                  |
| Stockholm (Swe)           | 12.865  | 13.532  | 13.620   | 13.729  | 13.699                                 |
| Swinoujscie (Pol)         | 2.312   | 2.877   | 2.516  | 2.947   | 3.004                                  |
| Ustka (Pol)               | 1.535   | —   | 1.720  | 2.180   | 2.379                                  |
| Vaasa (Fin)               | 1.128   | 1.828   | 1.772  | 2.131   | 2.060                                  |
| Visby (Swe)               | 1.986   | 2.278   | 2.547  | 2.713   | 2.533                                  |
| Warnemünde (Ger)          | —   | —   | 2.963  | 3.307   | 3.246                                  |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 13.** Orthometric height of the GPS stations of the Baltic Sea Level project, second campaign, 1993.4, derived from different sources (cases 1–5)

| Station name <sup>a</sup> | Orthometric height<br>(case 1) (Poutanen et al. 1999) | Orthometric height<br>(case 2) (Kakkuri 1995) | Orthometric height<br>(case 3) (Kakkuri and Poutanen 1997) | Orthometric height<br>(case 4) (Poutanen et al. 1999) | Orthometric height<br>(case 5) (EGG97) |
|---------------------------|---|---|--|---|--|
| Borkum (Ger)              | 4.578   | 4.433   | –  | 4.740   | 4.874                                  |
| Degerby (Fin)             | 1.681   | 1.708   | 1.791  | 1.745   | 1.906                                  |
| Furuögrund (Swe)          | 10.936  | 11.074  | 11.018   | 11.121  | 11.308                                 |
| Hamina (Fin)              | 1.624   | 1.710   | 1.693  | 1.818   | 1.982                                  |
| Hanko (Fin)               | 1.769   | 1.824   | 1.863  | 1.926   | 2.086                                  |
| Helgoland (Ger)           | 4.537   | –   | –  | 4.562   | 4.687                                  |
| Helsinki (Fin)            | 6.039   | 6.069   | 6.078  | 6.187   | 6.349                                  |
| Kemi (Fin)                | 1.266   | 1.355   | 1.442  | 1.562   | 1.742                                  |
| Klagshamn (Swe)           | 2.039   | 2.060   | 1.678  | 2.152   | 2.286                                  |
| Klaipeda (Lit)            | 28.209  | –   | –  | 28.323  | –                                      |
| Kronstadt (Rus)           | –   | –   | –  | –   | –                                      |
| List/Sylt (Ger)           | 4.159   | 3.930   | –  | 4.199   | 4.330                                  |
| Mäntyluoto (Fin)          | 2.484   | 2.533   | 2.547  | 2.608   | –                                      |
| Molas (Lit)               | 4.577   | –   | –  | 4.664   | 4.799                                  |
| Ölands N. U. (Swe)        | 4.122   | 3.906   | 4.127  | 4.230   | 4.375                                  |
| Raahe (Fin)               | 2.086   | 2.139   | 2.180  | 2.337   | 2.523                                  |
| Ratan (Swe)               | 1.500   | 1.660   | 1.567  | 1.698   | 1.881                                  |
| Shepelevo (Rus)           | –   | 4.211   | 4.262  | 4.317   | –                                      |
| Spikarna (Swe)            | 1.893   | 2.365   | 2.030  | 2.063   | 2.237                                  |
| Stockholm (Swe)           | 12.873  | 13.020  | 13.108   | 13.062  | 13.219                                 |
| Swinoujscie (Pol)         | 2.309   | 2.367   | 2.006  | 2.315   | 2.439                                  |
| Ustka (Pol)               | 1.532   | –   | 1.187  | 1.521   | 1.649                                  |
| Vaasa (Fin)               | 1.149   | 1.227   | 1.171  | 1.353   | 1.551                                  |
| Visby (Swe)               | 1.986   | 1.775   | 2.044  | 2.065   | 2.212                                  |
| Warnemünde (Ger)          | 21.291  | –   | 21.146   | 21.367  | 21.492                                 |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia**Table 14.** Orthometric height of the GPS stations of the Baltic Sea Level project, third campaign, 1997.4, derived from different sources (cases 1–5)

| Station name <sup>a</sup> | Orthometric height<br>(case 1) (Poutanen et al. 1999) | Orthometric height<br>(case 2) (Kakkuri 1995) | Orthometric height<br>(case 3) (Kakkuri and Poutanen 1997) | Orthometric height<br>(case 4) (Poutanen et al. 1999) | Orthometric height<br>(case 5) (Kakkuri 1995) |
|---------------------------|---|---|--|---|---|
| Borkum (Ger)              | 4.574   | 4.405   | –  | 4.712   | 4.846   |
| Degerby (Fin)             | 2.825   | 2.877   | 2.960  | 2.914   | 3.075   |
| Furuögrund (Swe)          | 10.972  | 11.108  | 11.052   | 11.155  | 11.342  |
| Hamina (Fin)              | 1.631   | 1.697   | 1.680  | 1.805   | 1.969   |
| Hanko (Fin)               | 5.118   | 5.173   | 5.212  | 5.275   | 5.435   |
| Helgoland (Ger)           | 4.531   | –   | –  | 4.610   | 4.735   |
| Helsinki (Fin)            | 6.420   | 6.455   | 6.464  | 6.573   | 6.735   |
| Kemi (Fin)                | 7.092   | 7.185   | 7.272  | 7.392   | 7.572   |
| Klagshamn (Swe)           | 2.038   | 2.099   | 1.717  | 2.191   | 2.325   |
| Klaipeda (Lit)            | 28.209  | –   | –  | 28.336  | –   |
| Kronstadt (Rus)           | –   | –   | –  | 4.739   | –   |
| List/Sylt (Ger)           | 4.155   | 3.916   | –  | 4.185   | 4.316   |
| Mäntyluoto (Fin)          | 2.509   | 2.562   | 2.576  | 2.637   | –   |
| Molas (Lit)               | 4.577   | –   | –  | 4.679   | 4.814   |
| Ölands N. U. (Swe)        | 4.127   | 3.917   | 4.138  | 4.241   | 4.386   |
| Raahe (Fin)               | 3.528   | 3.607   | 3.648  | 3.805   | 3.991   |
| Ratan (Swe)               | 1.535   | 1.691   | 1.598  | 1.729   | 1.912   |
| Shepelevo (Rus)           | –   | 4.282   | 4.333  | 4.388   | –   |
| Spikarna (Swe)            | 1.924   | 2.390   | 2.055  | 2.088   | 2.262   |
| Stockholm (Swe)           | 11.905  | 12.027  | 12.115   | 12.069  | 12.226  |
| Swinoujscie (Pol)         | –   | 6.681   | 6.320  | 6.629   | 6.753   |
| Ustka (Pol)               | 1.528   | –   | 1.181  | 1.515   | 1.643   |
| Vaasa (Fin)               | 1.180   | 1.275   | 1.219  | 1.401   | 1.599   |
| Visby (Swe)               | 1.992   | 1.771   | 2.040  | 2.061   | 2.208   |
| Warnemünde (Ger)          | 11.319  | –   | 11.089   | 11.310  | 11.435  |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 15.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$  free-air gravity potential  $\Delta W(\lambda, \phi, u)$  computed based on the orthometric heights of case 1 (Poutanen et al. 1999), and gauge value of geoid potential  $W_0$  for the epoch of 1990.8.  $W_0$  at Lisk (Ger) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\Delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 475.745   | 38 330.956   | 44.956   | 62 636 851.657   |
| Degerby (Fin)             | 62 609 712.120   | 27 124.194   | 16.643   | 62 636 852.957   |
| Furuögrund (Swe)          | 62 617 207.876   | 19 542.016   | 107.183  | 62 636 857.076   |
| Hamina (Fin)              | 62 610 584.084   | 26 255.428   | 15.897   | 62 636 855.409   |
| Hanko (Fin)               | 62 609 368.721   | 27 467.501   | 17.301   | 62 636 853.523   |
| Helgoland (Ger)           | 62 599 594.591   | 37 218.925   | 44.546   | 62 636 858.063   |
| Helsinki (Fin)            | 62 609 873.004   | 26 924.677   | 59.239   | 62 636 856.920   |
| Kemi (Fin)                | 62 618 386.248   | 18 455.996   | 12.240   | 62 636 854.484   |
| Klagshamn (Swe)           | 62 602 016.194   | 34 818.793   | 20.014   | 62 636 855.001   |
| Klaipeda (Lit)            | –  | –  | –  | –  |
| Kronstadt (Rus)           | –  | –  | –  | –  |
| List/Sylt (Ger)           | 47 933 936.649   | 111 867.236  | 23.866   | 48 045 827.752   |
| Mäntyluoto (Fin)          | 62 612 227.796   | 24 604.331   | 24.226   | 62 636 856.353   |
| Molas (Lit)               | –  | –  | –  | –  |
| Ölands N. U. (Swe)        | 62 605 215.375   | 31 601.731   | 40.426   | 62 636 857.531   |
| Raahe (Fin)               | 62 616 906.611   | 19 924.527   | 22.483   | 62 636 853.351   |
| Ratan (Swe)               | 62 615 931.246   | 20 910.717   | 14.497   | 62 636 856.460   |
| Shepelevo (Rus)           | –  | –  | –  | –  |
| Spikarna (Swe)            | 62 613 442.333   | 23 395.357   | 18.384   | 62 636 856.074   |
| Stockholm (Swe)           | 62608 433.790  | 28 295.608   | 126.314  | 62 636 855.712   |
| Swinoujscie (Pol)         | 62 599 134.600   | 37 698.887   | 22.690   | 62 636 856.177   |
| Ustka (Pol)               | 62 600 357.058   | 36 479.270   | 15.066   | 62 636 851.393   |
| > Vaasa (Fin)             | 62 614 579.338   | 22 265.876   | 11.078   | 62 636 856.292   |
| Visby (Swe)               | 62 605 700.835   | 31 135.657   | 19.497   | 62 636 855.989   |
| Warnemünde (Ger)          | 62 599 616.182   | 37 213.250   | –  | –  |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 855.285   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.4268   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 16.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u)$  computed based on the orthometric heights of case 2 (Kakkuri 1995), and gauge value of geoid potential  $W_0$  for the epoch of 1990.8.  $W_0$  at Lisk (Ger) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\Delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 475.745   | 38 330.956   | 47.527   | 62 636 854.228   |
| Degerby (Fin)             | 62 609 712.120   | 27 124.194   | 16.436   | 62 636 852.751   |
| Furuögrund (Swe)          | 62 617 207.876   | 19 542.016   | 107.999  | 62 636 857.891   |
| Hamina (Fin)              | 62 610 584.084   | 26 255.428   | 16.595   | 62 636 856.106   |
| Hanko (Fin)               | 62 609 368.721   | 27 467.501   | 17.625   | 62 636 853.847   |
| Helgoland (Ger)           | 62 599 594.591   | 37 218.925   | –  | –  |
| Helsinki (Fin)            | 62 609 873.004   | 26 924.677   | 59.082   | 62 636 856.763   |
| Kemi (Fin)                | 62 618 386.248   | 18 455.996   | 12.780   | 62 636 855.024   |
| Klagshamn (Swe)           | 62 602 016.194   | 34 818.793   | 19.572   | 62 636 854.559   |
| Klaipeda (Lit)            | –  | –  | –  | –  |
| Kronstadt (Rus)           | –  | –  | –  | –  |
| List/Sylt (Ger)           | 47 933 936.649   | 111 867.236  | 22.926   | 48 045 826.811   |
| Mäntyluoto (Fin)          | 62 612 227.796   | 24 604.331   | 24.746   | 62 636 856.873   |
| Molas (Lit)               | –  | –  | –  | –  |
| Ölands N. U. (Swe)        | 62 605 215.375   | 31 601.731   | 37.795   | 62 636 854.900   |
| Raahe (Fin)               | 62 616 906.611   | 19 924.257   | 23.858   | 62 636 854.726   |
| Ratan (Swe)               | 62 615 931.246   | 20 910.717   | 17.257   | 62 636 859.220   |
| Shepelevo (Rus)           | –  | –  | –  | –  |
| Spikarna (Swe)            | 62 613 442.333   | 23 395.357   | 22.784   | 62 636 860.473   |
| Stockholm (Swe)           | 62 608 433.790   | 28 295.608   | 126.608  | 62 636 856.006   |
| Swinoujscie (Pol)         | 62 599 134.600   | 37 698.889   | 22.406   | 62 636 855.893   |
| Ustka (Pol)               | 62 600 357.058   | 36 479.270   | –  | –  |
| Vaasa (Fin)               | 62 614 579.338   | 22 265.876   | 11.500   | 62 636 856.714   |
| Visby (Swe)               | 62 605 700.835   | 31 135.657   | 16.326   | 62 636 852.818   |
| Warnemünde (Ger)          | 62 599 616.182   | 37 213.250   | –  | –  |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 6 2636 855.811   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.5056   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 17.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u)$  computed based on orthometric heights of case 3 (Kakkuri and Poutanen 1997), and gauge value of geoid potential  $W_0$  for the epoch of 1990.8.  $W_0$  at Ustka (Pol) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\Delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 475.745   | 38 330.956   | –  | –  |
| Degerby (Fin)             | 62 609 712.120   | 27 124.194   | 17.251   | 62 636 853.566   |
| Furuögrund (Swe)          | 62 617 207.876   | 19 542.016   | 107.449  | 62 636 857.341   |
| Hamina (Fin)              | 62 610 584.084   | 26 255.428   | 16.428   | 62 636 855.939   |
| Hanko (Fin)               | 62 609 368.721   | 27 467.501   | 18.008   | 62 636 854.230   |
| Helgoland (Ger)           | 62 599 594.591   | 37 218.925   | –  | –  |
| Helsinki (Fin)            | 62 609 873.004   | 26 924.677   | 59.170   | 62 636 856.852   |
| Kemi (Fin)                | 62 618 386.248   | 18 455.996   | 13.634   | 62 636 855.879   |
| Klagshamn (Swe)           | 62 602 016.194   | 34 818.793   | 15.822   | 62 636 850.809   |
| Klaipeda (Lit)            | –  | –  | –  | –  |
| Kronstadt (Rus)           | –  | –  | –  | –  |
| List/Sylt (Ger)           | 47 933 936.649   | 111 867.236  | –  | –  |
| Mäntyluoto (Fin)          | 62 612 227.796   | 24 604.331   | 24.884   | 62 636 857.010   |
| Molas (Lit)               | –  | –  | –  | –  |
| Ölands N. U. (Swe)        | 62 605 215.375   | 31 601.731   | 39.694   | 62 636 857.070   |
| Raahe (Fin)               | 62 616 906.611   | 19 924.257   | 24.261   | 62 636 855.129   |
| Ratan (Swe)               | 62 615 931.246   | 20 910.717   | 16.344   | 62 636 858.307   |
| Shepelevo (Rus)           | –  | –  | –  | –  |
| Spikarna (Swe)            | 62 613 442.333   | 23 395.357   | 19.494   | 62 636 857.184   |
| Stockholm (Swe)           | 62 608 433.790   | 28 295.608   | 127.472  | 62 636 856.870   |
| Swinoujście (Pol)         | 62 599 134.600   | 37 698.887   | 18.863   | 62 636 852.350   |
| Ustka (Pol)               | 62 600 357.058   | 36 479.270   | 11.061   | 62 636 847.389   |
| Vaasa (Fin)               | 62 614 579.338   | 22 265.876   | 10.950   | 62 636 856.164   |
| Visby (Swe)               | 62 605 700.835   | 31 135.657   | 18.967   | 62 636 855.459   |
| Warnemünde (Ger)          | 62 599 616.182   | 37 213.250   | 22.367   | 62 636 851.798   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 855.409   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.5227   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

dex.html projects). We use WGD2000 in the mean-tide system as the reference ellipsoid (cf. Grafarend and Ardalan 1999). In other words, we perform all the computations in the mean-tide/permanent-tide system.

We obtained the orthometric heights of GPS stations (Tables 12–14) from five different sources, namely Poutanen et al. (1999), Kakkuri (1995), Kakkuri and Poutanen (1997), Poutanen et al. (1999) and Kakkuri (2000), respectively. Consequently, we performed the computations under the five cases defined below.

*Case 1:* the orthometric heights from Poutanen et al. (1999)

*Case 2:* the orthometric heights from Kakkuri (1995)

*Case 3:* the orthometric heights from Kakkuri and Poutanen (1997)

*Case 4:* the orthometric heights from Poutanen et al. (1999)

*Case 5:* the heights from Kakkuri (2000)

In case 1, the orthometric heights  $H_0$  are given in their respective national height systems, measured directly by precise levelling to tide gauges, while for cases 2–5 orthogon heights  $H_0$  are derived from GPS ellipsoidal height  $h$  and different geoid solutions  $N$  proposed for the Baltic Sea. Recall that, neglecting the curvature of the plumb line, the following relation holds:

$$H_0 \doteq h - N \quad (17)$$

It should be mentioned that the difference between the orthometric heights of cases 2–5 is mainly due to different geoid solutions and the application of different reductions to obtain the mean geoid via polynomial fitting, and the use of different starting latitudes [see e.g. Vermeer (1995) for more details].

In the following section, the results of  $w_0$  computations for each of the above-mentioned cases will be presented.

#### 4 Baltic Sea Level project: first campaign, 1990.8

Here we present the results of  $w_0$  computation based on the GPS observation of the Baltic Sea Level project, first campaign, in the five cases introduced previously. Tables 15–19 present the gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$  and free-air gravity potential  $\Delta W(\lambda, \phi, u)$  computed based on the orthometric heights of cases 1–5. A summary of the mean values of  $w_0$  derived from the different cases and the standard deviation of those mean values is given in Table 20.

Now let us summarise the results we obtain in the five different cases for the geoid potential value  $w_0$  based on GPS observations of the Baltic Sea Level project, first campaign, epoch 1990.8, shown in Table 20. From a review of Table 20 following conclusions can be made.

1. The most consistent results correspond to case 4.

**Table 18.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u)$  computed based on orthometric heights of case 4 (Poutanen et al. 1999), and gauge value of geoid potential  $W_0$  for the epoch of 1990.8.  $W_0$  values at Ustka (Pol) and Lisk (Ger) are outliers, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\Delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 475.745   | 38 330.956   | 50.540   | 62 636 857.241   |
| Degerby (Fin)             | 62 609 712.120   | 27 124.194   | 16.800   | 62 636 853.114   |
| Furuögrund (Swe)          | 62 617 207.876   | 19 542.016   | 108.460  | 62 636 858.352   |
| Hamina (Fin)              | 62 610 584.084   | 26 255.428   | 17.655   | 62 636 857.167   |
| Hanko (Fin)               | 62 609 368.721   | 27 467.501   | 18.627   | 62 636 854.849   |
| Helgoland (Ger)           | 62 599 594.591   | 37 218.925   | 44.782   | 62 636 858.298   |
| Helsinki (Fin)            | 62 609 873.004   | 26 924.677   | 60.241   | 62 636 857.922   |
| Kemi (Fin)                | 62 618 386.248   | 18 455.996   | 14.813   | 62 636 857.058   |
| Klagshamn (Swe)           | 62 602 016.194   | 34 818.793   | 20.672   | 62 636 855.462   |
| Klaipeda (Lit)            | —  | —  | —  | —  |
| Kronstadt (Rus)           | —  | —  | —  | —  |
| List/Sylt (Ger)           | 47 933 936.649   | 111 867.236  | 24.469   | 48 045 828.354   |
| Mäntyluoto (Fin)          | 62 612 227.796   | 24 604.331   | 25.483   | 62 636 857.609   |
| Molas (Lit)               | —  | —  | —  | —  |
| Ölands N. U. (Swe)        | 62 605 215.375   | 31 601.731   | 40.975   | 62 636 858.081   |
| Raahe (Fin)               | 62 616 906.611   | 19 924.257   | 25.803   | 62 636 856.671   |
| Ratan (Swe)               | 62 615 931.246   | 20 910.717   | 17.631   | 62 636 859.594   |
| Shepelevo (Rus)           | —  | —  | —  | —  |
| Spikarna (Swe)            | 62 613 442.333   | 23 395.357   | 19.818   | 62 636 857.508   |
| Stockholm (Swe)           | 62 608 433.790   | 28 295.608   | 127.021  | 62 636 856.419   |
| Swinoujscie (Pol)         | 62 599 134.600   | 37 698.887   | 21.895   | 62 636 855.382   |
| Ustka (Pol)               | 62 600 357.058   | 36 479.270   | 14.339   | 62 636 850.667   |
| Vaasa (Fin)               | 62 614 579.338   | 22 265.876   | 12.738   | 62 636 857.951   |
| Visby (Swe)               | 62 605 700.835   | 31 135.657   | 19.173   | 62 636 855.665   |
| Warnemünde (Ger)          | 62 599 616.182   | 37 213.250   | 24.536   | 62 636 853.967   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 856.753   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.3780   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 19.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u)$  based on orthometric heights of case 5 (Kakkuri 2000), and gauge value of geoid potential  $W_0$  for the epoch of 1990.8.  $W_0$  values at Ustka (Pol) and Lisk (Ger) are outliers, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\Delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 475.745   | 38 330.956   | 51.855   | 62 636 858.556   |
| Degerby (Fin)             | 62 609 712.120   | 27 124.194   | 18.380   | 62 636 854.695   |
| Furuögrund (Swe)          | 62 617 207.876   | 19 542.016   | 110.297  | 62 636 860.189   |
| Hamina (Fin)              | 62 610 584.084   | 26 255.428   | 19.265   | 62 636 858.777   |
| Hanko (Fin)               | 62 609 368.721   | 27 467.501   | 20.198   | 62 636 856.420   |
| Helgoland (Ger)           | 62 599 594.591   | 37 218.925   | 46.008   | 62 636 859.525   |
| Helsinki (Fin)            | 62 609 873.004   | 26 924.677   | 61.831   | 62 636 859.513   |
| Kemi (Fin)                | 62 618 386.248   | 18 455.996   | 16.581   | 62 636 858.826   |
| Kalgshamn (Swe)           | 62 602 016.194   | 34 818.793   | 21.790   | 62 636 856.777   |
| Klaipeda (Lit)            | —  | —  | —  | —  |
| Kronstadt (Rus)           | —  | —  | —  | —  |
| List/Sylt (Ger)           | 47 933 936.649   | 111 867.236  | 25.220   | 48 045 829.106   |
| Mäntyluoto (Fin)          | 62 612 227.796   | 24 604.331   | —  | —  |
| Molas (Lit)               | —  | —  | —  | —  |
| Ölands N. U. (Swe)        | 62 605 215.375   | 31 601.731   | 42.399   | 62 636 859.504   |
| Raahe (Fin)               | 62 616 906.611   | 19 924.257   | 27.630   | 62 636 858.498   |
| Ratan (Swe)               | 62 615 931.246   | 20 910.717   | 19.428   | 62 636 861.391   |
| Shepelevo (Rus)           | —  | —  | —  | —  |
| Spikarna (Swe)            | 62 613 442.333   | 23 395.357   | 21.527   | 62 636 859.216   |
| Stockholm (Swe)           | 62 608 433.790   | 28 295.608   | 128.562  | 62 636 857.960   |
| Swinoujscie (Pol)         | 62 599 134.600   | 37 698.887   | 23.112   | 62 636 856.599   |
| Ustka (Pol)               | 62 600 357.058   | 36 479.270   | 15.596   | 62 636 851.923   |
| Vaasa (Fin)               | 62 614 579.338   | 22 265.876   | 14.682   | 62 636 859.896   |
| Visby (Swe)               | 62 605 700.835   | 31 135.657   | 20.616   | 62 636 857.108   |
| Warnemünde (Ger)          | 62 599 616.182   | 37 213.250   | 25.763   | 62 636 855.194   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 858.258   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.4218   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

2. The results of case 5 show a shift of about  $2.4 \text{ m}^2/\text{s}^2$  with respect to the mean value of  $w_0$  calculated from the other four cases.
3. The weighted mean of the geoid potential value  $w_0$  based on the GPS observation of the Baltic Sea Level project, first campaign, and orthometric heights of all cases is  $62\,636\,856.434 \pm 0.558 \text{ m}^2/\text{s}^2$ , and based on the orthometric heights of cases 1–4 is  $62\,636\,855.922 \pm 0.366 \text{ m}^2/\text{s}^2$ .

### 5 Baltic Sea Level project: second campaign, 1993.4

In this section we will review the results of  $w_0$  computation based on the GPS observations of the Baltic Sea Level project, second campaign, in the five

**Table 20.** Summary of  $w_0$  values computed from the GPS observations of the Baltic Sea Level project, first campaign, 1990.8, for cases 1–5

| Case | $w_0 \text{ (m}^2/\text{s}^2)$ | Standard deviation ( $w_0$ ) ( $\text{m}^2/\text{s}^2$ ) |
|------|--------------------------------|--|
| 1    | 62 636 855.285                 | 0.4268   |
| 2    | 62 636 855.811                 | 0.5056   |
| 3    | 62 636 855.409                 | 0.5227   |
| 4    | 62 636 856.753                 | 0.3780   |
| 5    | 62 636 858.258                 | 0.4218   |

cases introduced previously. Tables 21–25 present the gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$  and free-air gravity potential  $\Delta W(\lambda, \phi, u)$  computed based on the orthometric heights of cases 1–5. A summary of the mean values derived from the different cases and the standard deviation of those mean values is given in Table 26.

Now let us summarise the results obtained for  $w_0$  in the five different cases, based on GPS observations of the Baltic Sea Level project, second campaign, epoch 1993.4, shown in Table 26. From a review of Table 26 the following conclusions can be made.

1. The results of the second campaign are in general more accurate than those of the first campaign, due to the more accurate GPS observation made under more favourable ionospheric conditions, as is explained by Kakkuri (1995).
2. The most consistent results correspond to case 4.
3. The results of case 5 show a shift of about  $2.5 \text{ m}^2/\text{s}^2$  with respect to the mean value of  $w_0$  calculated from cases 1–4.
4. The mean of the geoid potential value  $w_0$  based on the GPS observation of the Baltic Sea Level project, second campaign, and the orthometric heights of all cases is  $62\,636\,855.926 \pm 0.536 \text{ m}^2/\text{s}^2$ , and based on the orthometric heights of cases 1–4 is  $62\,636\,855.515 \pm 0.385 \text{ m}^2/\text{s}^2$ .

**Table 21.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 1 (Poutanen et al. 1999), and gauge value of geoid potential  $W_0$  for the epoch of 1993.4

| Station name <sup>a</sup> | Gravitational potential $U(\lambda, \phi, u) \text{ (m}^2/\text{s}^2)$ | Centrifugal potential $V(\lambda, \phi, u) \text{ (m}^2/\text{s}^2)$ | Free-air reduction $\delta W(\lambda, \phi, u) \text{ (m}^2/\text{s}^2)$ | Gauge value $W_0 \text{ (m}^2/\text{s}^2)$ |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 479.287   | 38 330.949   | 44.927   | 62 636 855.163                             |
| Degerby (Fin)             | 62 609 711.662   | 27 124.193   | 16.505   | 62 636 852.360                             |
| Furuögrund (Swe)          | 62 617 205.887   | 19 542.017   | 107.419  | 62 636 855.323                             |
| Hamina (Fin)              | 62 610 584.027   | 26 255.426   | 15.946   | 62 636 855.400                             |
| Hanko (Fin)               | 62 609 367.961   | 27 467.499   | 17.370   | 62 636 852.829                             |
| Helgoland (Ger)           | 62 599 593.587   | 37 218.924   | 44.526   | 62 636 857.037                             |
| Helsinki (Fin)            | 62 609 872.036   | 26 924.677   | 59.298   | 62 636 856.010                             |
| Kemi (Fin)                | 62 618 386.991   | 18 455.995   | 12.436   | 62 636 855.423                             |
| Klagshamn (Swe)           | 62 602 014.353   | 34 818.793   | 20.014   | 62 636 853.160                             |
| Klaipeda (Lit)            | 62 602 168.578   | 34 409.334   | 276.885  | 62 636 854.797                             |
| Kronstadt (Rus)           | –  | –  | –  | –  |
| List/Sylt (Ger)           | 62 601 101.969   | 35 713.293   | 40.821   | 62 636 856.083                             |
| Mäntyluoto (fin)          | 62 612 227.594   | 24 604.328   | 24.393   | 62 636 856.316                             |
| Molas (Lit)               | 62 602 356.820   | 34 452.784   | 44.926   | 62 636 854.530                             |
| Ölands N. U. (Swe)        | 62 605 214.319   | 31 601.730   | 40.465   | 62 636 856.514                             |
| Raahe (Fin)               | 62 616 895.232   | 19 938.331   | 20.489   | 62 636 854.053                             |
| Ratan (Swe)               | 62 615 931.059   | 20 910.716   | 14.733   | 62 636 856.508                             |
| Shepelevo (Rus)           | 62 609 585.155   | 27 231.095   | –  | –  |
| Spikarna (Swe)            | 62 613 440.944   | 23 395.356   | 18.591   | 62 636 854.891                             |
| Stockholm (Swe)           | 62 608 431.963   | 28 295.609   | 126.392  | 62 636 853.964                             |
| Swinoujście (Pol)         | 62 599 133.126   | 37 698.887   | 22.661   | 62 636 854.674                             |
| Ustka (Pol)               | 62 600 356.012   | 36 479.269   | 15.036   | 62 636 850.317                             |
| Vaasa (Fin)               | 62 614 578.263   | 22 265.875   | 0.284  | 62 636 855.423                             |
| Visby (Swe)               | 62 605 698.773   | 31 135.657   | 19.497   | 62 636 853.927                             |
| Warnemünde (Ger)          | 62 599 433.944   | 37 210.901   | 208.957  | 62 636 853.802                             |
|                           |  |  | Mean ( $\text{m}^2/\text{s}^2$ )   | 62 636 854.718                             |
|                           |  |  | Standard deviation ( $\text{m}^2/\text{s}^2$ )                           | 0.3248                                     |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 22.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u, H_0)$ , computed based on orthometric heights of case 2 (Kakkuri 1995) and gauge value of geoid potential  $W_0$  for the epoch of 1993.4

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 479.287   | 38 330.949   | 43.504   | 62 636 853.740   |
| Degerby (Fin)             | 62 609 711.662   | 27 124.193   | 16.770   | 62 636 852.625   |
| Furuögrund (Swe)          | 62 617 205.887   | 19 542.017   | 108.775  | 62 636 856.678   |
| Hamina (Fin)              | 62 610 584.027   | 26 255.426   | 16.791   | 62 636 856.244   |
| Hanko (Fin)               | 62 609 367.961   | 27 467.499   | 17.910   | 62 636 853.369   |
| Helgoland (Ger)           | 62 599 593.587   | 37 218.924   | —  | —  |
| Helsinki (Fin)            | 62 609 872.036   | 26 924.677   | 59.592   | 62 636 856.305   |
| Kemi (Fin)                | 62 618 386.991   | 18 455.995   | 13.310   | 62 636 856.297   |
| Klagshamn (Swe)           | 62 602 014.353   | 34 818.793   | 20.220   | 62 636 853.366   |
| Klaipeda (Lit)            | 62 602 168.578   | 34 409.334   | —  | —  |
| Kronstadt (Rus)           | —  | —  | —  | —  |
| List/Sylt (Ger)           | 62 601 101.969   | 35 713.293   | 38.574   | 62 636 853.836   |
| Mäntyluoto (Fin)          | 62 612 227.594   | 24 604.328   | 24.874   | 62 636 856.797   |
| Molas (Lit)               | 62 602 356.820   | 34 452.784   | —  | —  |
| Ölands N. U. (Swe)        | 62 605 214.319   | 31 601.730   | 38.345   | 62 636 854.393   |
| Raahe (Fin)               | 62 616 895.232   | 19 938.331   | 21.010   | 62 636 854.573   |
| Ratan (Swe)               | 62 615 931.059   | 20 910.716   | 16.305   | 62 636 858.079   |
| Shepelevo (Rus)           | 62 609 585.155   | 27 231.095   | 41.348   | 62 636 857.598   |
| Spikarna (Swe)            | 62 613 440.944   | 23 395.356   | 23.226   | 62 636 859.526   |
| Stockholm (Swe)           | 62 608 431.963   | 28 295.609   | 127.836  | 62 636 855.407   |
| Swinoujscie (Pol)         | 62 599 133.126   | 37 698.887   | 23.230   | 62 636 855.243   |
| Ustka (Pol)               | 62 600 356.012   | 36 479.269   | —  | —  |
| Vaasa (Fin)               | 62 614 578.263   | 22 265.875   | 12.050   | 62 636 856.189   |
| Visby (Swe)               | 62 605 698.773   | 31 135.657   | 17.426   | 62 636 851.855   |
| Warnemünde (Ger)          | 62 599 433.944   | 37 210.901   | —  | —  |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 855.375   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.4557   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia**Table 23.** Gravitational potential  $U(\lambda, \phi, \eta)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 3 (Kakkuri and Poutanen 1997), and gauge value of geoid potential  $W_0$  for the epoch of 1993.4.  $W_0$  values at Ustka (Pol) and Klagshamn (Swe) are outliers, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 479.287   | 38 330.949   | —  | —  |
| Degerby (Fin)             | 62 609 711.662   | 27 124.193   | 17.585   | 62 636 853.440   |
| Furuögrund (Swe)          | 62 617 205.887   | 19 542.017   | 108.225  | 62 636 856.128   |
| Hamina (Fin)              | 62 610 584.027   | 26 255.426   | 16.624   | 62 636 856.007   |
| Hanko (Fin)               | 62 609 367.961   | 27 467.499   | 18.293   | 62 636 853.752   |
| Helgoland (Ger)           | 62 599 593.587   | 37 218.924   | —  | —  |
| Helsinki (Fin)            | 62 609 872.036   | 26 924.677   | 59.681   | 62 636 856.393   |
| Kemi (Fin)                | 62 618 386.991   | 18 455.995   | 14.165   | 62 636 857.151   |
| Klagshamn (Swe)           | 62 602 014.353   | 34 818.793   | 16.470   | 62 636 859.616   |
| Klaipeda (Lit)            | 62 602 168.578   | 34 409.334   | —  | —  |
| Kronstadt (Rus)           | —  | —  | —  | —  |
| List/Sylt (Ger)           | 62 601 101.969   | 35 713.293   | —  | —  |
| Mäntyluoto (fin)          | 62 612 227.594   | 24 604.328   | 25.012   | 62 636 856.934   |
| Molas (Lit)               | 62 602 356.820   | 34 452.784   | —  | —  |
| Ölands N. U. (Swe)        | 62 605 214.319   | 31 601.730   | 40.514   | 62 636 856.563   |
| Raahe (Fin)               | 62 616 895.232   | 19 938.331   | 21.413   | 62 636 854.976   |
| Ratan (Swe)               | 62 615 931.059   | 20 910.716   | 15.391   | 62 636 857.166   |
| Shepelevo (Rus)           | 62 609 585.155   | 27 231.095   | 41.848   | 62 636 858.099   |
| Spikarna (Swe)            | 62 613 440.944   | 23 395.356   | 19.936   | 62 636 856.236   |
| Stockholm (Swe)           | 62 608 431.963   | 28 295.609   | 128.700  | 62 636 856.271   |
| Swinoujscie (Pol)         | 62 599 133.126   | 37 698.887   | 19.687   | 62 636 851.700   |
| Ustka (Pol)               | 62 600 356.012   | 36 479.269   | 11.650   | 62 636 846.931   |
| Vaasa (Fin)               | 62 614 578.263   | 22 265.875   | 11.500   | 62 636 855.639   |
| Visby (Swe)               | 62 605 698.773   | 31 135.657   | 20.066   | 62 636 854.496   |
| Warnemünde (Ger)          | 62 599 433.944   | 37 210.901   | 207.533  | 62 636 852.379   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 855.494   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.4325   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 24.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , based on orthometric heights of case 4 (Poutanen et al. 1997), and gauge value of geoid potential  $W_0$  for the epoch of 1993.4.  $W_0$  at Ustka (Pol) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 479.287   | 38 330.949   | 46.517   | 62 636 856.753   |
| Degerby (Fin)             | 62 609 711.662   | 27 124.193   | 17.134   | 62 636 852.988   |
| Furuögrund (Swe)          | 62 617 205.887   | 19 542.017   | 109.236  | 62 636 857.140   |
| Hamina (Fin)              | 62 610 584.027   | 26 255.426   | 17.851   | 62 636 857.305   |
| Hanko (Fin)               | 62 609 367.961   | 27 467.499   | 18.911   | 62 636 854.371   |
| Helgoland (Ger)           | 62 599 593.587   | 37 218.924   | 44.772   | 62 636 857.282   |
| Helsinki (Fin)            | 62 609 872.036   | 26 924.677   | 60.751   | 62 636 857.463   |
| Kemi (Fin)                | 62 618 386.991   | 18 455.995   | 15.344   | 62 636 858.330   |
| Klagshamn (Swe)           | 62 602 014.353   | 34 818.793   | 21.123   | 62 636 854.369   |
| Klaipeda (Lit)            | 62 602 168.578   | 34 409.334   | 278.004  | 62 636 855.916   |
| Kronstadt (Rus)           | —  | —  | —  | —  |
| List/Sylt (Ger)           | 62 601 101.969   | 35 713.293   | 41.214   | 62 636 856.476   |
| Mäntyluoto (Fin)          | 62 612 227.594   | 24 604.328   | 25.611   | 62 636 857.533   |
| Molas (Lit)               | 62 602 356.820   | 34 452.784   | 45.779   | 62 636 855.384   |
| Ölands N. U. (Swe)        | 62 605 214.319   | 31 601.730   | 41.525   | 62 636 857.574   |
| Raahe (Fin)               | 62 616 895.232   | 19 938.331   | 22.955   | 62 636 856.518   |
| Ratan (Swe)               | 62 615 931.059   | 20 910.716   | 16.678   | 62 636 858.453   |
| Shepelevo (Rus)           | 62 609 585.155   | 27 231.095   | 42.389   | 62 636 858.639   |
| Spikarna (Swe)            | 62 613 440.944   | 23 395.356   | 20.260   | 62 636 856.560   |
| Stockholm (Swe)           | 62 608 431.963   | 28 295.609   | 128.248  | 62 636 855.820   |
| Swinoujscie (Pol)         | 62 599 133.126   | 37 698.887   | 22.720   | 62 636 854.733   |
| Ustka (Pol)               | 62 600 356.012   | 36 479.269   | 14.928   | 62 636 840.210   |
| Vaasa (Fin)               | 62 614 578.263   | 22 265.875   | 13.288   | 62 636 857.426   |
| Visby (Swe)               | 62 605 698.773   | 31 135.657   | 20.273   | 62 636 854.702   |
| Warnemünde (Ger)          | 62 599 433.944   | 37 210.901   | 209.702  | 62 636 854.548   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 856.356   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.3177   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia**Table 25.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 5 (Kakkuri 2000), and gauge value of geoid potential  $W_0$  for the epoch of 1993.4.  $W_0$  at Ustka (Pol) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 479.287   | 38 330.949   | 47.832   | 62 636 858.068   |
| Degerby (Fin)             | 62 609 711.662   | 27 124.193   | 18.714   | 62 636 854.569   |
| Furuögrund (Swe)          | 62 617 205.887   | 19 542.017   | 111.073  | 62 636 858.977   |
| Hamina (Fin)              | 62 610 584.027   | 26 255.426   | 19.462   | 62 636 858.915   |
| Hanko (Fin)               | 62 609 367.961   | 27 467.499   | 20.485   | 62 636 855.942   |
| Helgoland (Ger)           | 62 599 593.587   | 37 218.924   | 45.999   | 62 636 858.509   |
| Helsinki (Fin)            | 62 609 872.036   | 26 924.677   | 62.342   | 62 636 859.054   |
| Kemi (Fin)                | 62 618 386.991   | 18 455.995   | 17.112   | 62 636 860.098   |
| Klagshamn (Swe)           | 62 602 014.353   | 34 818.793   | 22.438   | 62 636 855.584   |
| Klaipeda (Lit)            | 62 602 168.578   | 34 409.334   | —  | —  |
| Kronstadt (Rus)           | —  | —  | —  | —  |
| List/Sylt (Ger)           | 62 601 101.969   | 35 713.293   | 42.500   | 62 636 857.762   |
| Mäntyluoto (Fin)          | 62 612 227.594   | 24 604.328   | —  | —  |
| Molas (Lit)               | 62 602 356.820   | 34 452.784   | 47.105   | 62 636 856.709   |
| Ölands N. U. (Swe)        | 62 605 214.319   | 31 601.730   | 42.949   | 62 636 858.998   |
| Raahe (Fin)               | 62 616 895.232   | 19 938.331   | 24.782   | 62 636 858.345   |
| Ratan (Swe)               | 62 615 931.059   | 20 910.716   | 18.475   | 62 636 860.250   |
| Shepelevo (Rus)           | 62 609 585.155   | 27 231.095   | —  | —  |
| Spikarna (Swe)            | 62 613 440.944   | 23 395.356   | 21.969   | 62 636 858.269   |
| Stockholm (Swe)           | 62 608 431.963   | 28 295.609   | 129.789  | 62 636 857.361   |
| Swinoujscie (Pol)         | 62 599 133.126   | 37 698.887   | 23.936   | 62 636 855.950   |
| Ustka (Pol)               | 62 600 356.012   | 36 479.269   | 16.184   | 62 636 841.466   |
| Vaasa (Fin)               | 62 614 578.263   | 22 265.875   | 15.232   | 62 636 859.371   |
| Visby (Swe)               | 62 605 698.773   | 31 135.657   | 21.716   | 62 636 856.145   |
| Warnemünde (Ger)          | 62 599 433.944   | 37 210.901   | 210.929  | 62 636 855.775   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 857.733   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.3177   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 26.** Summary of  $w_0$  values computed from the GPS observations of the Baltic Sea Level project second campaign, 1993.4, for cases 1–5

| Case | $w_0$ (m <sup>2</sup> /s <sup>2</sup> ) | Standard deviation ( $w_0$ ) (m <sup>2</sup> /s <sup>2</sup> ) |
|------|---|--|
| 1    | 62 636 854.718                          | 0.3248   |
| 2    | 62 636 855.375                          | 0.4557   |
| 3    | 62 636 855.494                          | 0.4325   |
| 4    | 62 636 856.356                          | 0.3177   |
| 5    | 62 636 857.733                          | 0.3658   |

## 6 Baltic Sea Level project: third campaign, 1997.4

Here we present the results of  $w_0$  computation based on the GPS observations of the Baltic Sea Level project, third campaign, in the five cases introduced previously. Tables 27–31 present gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$  and free-air gravity potential  $\Delta W(\lambda, \phi, u)$  computed based on the orthometric heights of cases 1–5. A summary of the mean values derived from the different cases and the standard deviation of those mean values is given in Table 32.

Now let us summarise those results we obtained in the five different cases for  $w_0$ , based on GPS observation of the Baltic Sea Level project, third campaign, epoch 1997.4, shown in Table 32. From a review of Table 32 the following conclusions can be made.

1. The results of  $w_0$  computations based on the data of the third campaign are in general more accurate than the results we obtained for the first and the second campaigns, which indicates improvement in the accuracy of the GPS observations.
2. The most consistent results correspond to the heights of case 4.
3. The results of case 5 shows a shift of about 2.3 m<sup>2</sup>/s<sup>2</sup> with respect to mean value of  $w_0$  calculated from the other four cases.
4. The weighted mean of the geoid potential value  $w_0$  based on the GPS observations of the Baltic Sea Level project, third campaign, and orthometric heights of all cases is  $62\,636\,856.248 \pm 0.525$  m<sup>2</sup>/s<sup>2</sup>, and based on the orthometric heights of cases 1–4 is  $62\,636\,855.804 \pm 0.378$  m<sup>2</sup>/s<sup>2</sup>.
5. Finally, from a review of Tables 20, 26 and 32 we can conclude with the value of  $w_0 = 62\,636\,855.75 \pm 0.21$  m<sup>2</sup>/s<sup>2</sup> as our best estimate of the geoid potential value.

## 7 Time derivative of $W_0$

In the previous section we presented the value  $w_0 = 62\,636\,855.75 \pm 0.21$  m<sup>2</sup>/s<sup>2</sup>, from the analysis of  $w_0$  values computed in five cases and three campaigns of GPS observations of the Baltic Sea Level project.

**Table 27.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 1 (Poutanen et al. 1999), and gauge value of geoid potential  $W_0$  for the epoch of 1997.4.  $W_0$  at Ustka (Pol) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential $U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential $V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction $\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value $W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|---|---|---|---|
| Borkum (Ger)              | 62 598 480.049  | 38 330.947  | 44.888  | 62 636 855.884                                      |
| Degerby (Fin)             | 62 609 699.381  | 27 124.102  | 27.738  | 62 636 852.221                                      |
| Furuögrund (Swe)          | 62 617 205.722  | 19 542.016  | 107.773   | 62 636 855.551                                      |
| Hamina (Fin)              | 62 610 584.135  | 26 255.426  | 16.015  | 62 636 855.576                                      |
| Hanko (Fin)               | 62 609 334.840  | 27 467.913  | 50.254  | 62 636 853.007                                      |
| Helgoland (Ger)           | 62 599 593.036  | 37 218.924  | 44.468  | 62 636 856.427                                      |
| Helsinki (Fin)            | 62 609 868.216  | 26 924.835  | 63.039  | 62 636 856.090                                      |
| Kemi (Fin)                | 62 618 330.677  | 18 455.223  | 69.665  | 62 636 855.565                                      |
| Klagshamn (Swe)           | 62 602 014.413  | 34 818.792  | 20.004  | 62 636 853.208                                      |
| Klaipeda (Lit)            | 62 602 168.314  | 34 409.333  | 276.885   | 62 636 854.532                                      |
| Kronstadt (Rus)           | 62 609 616.854  | 27 194.595  | –   | –   |
| List/Sylt (Ger)           | 62 601 102.613  | 35 713.291  | 40.782  | 62 636 856.686                                      |
| Mäntyluoto (fin)          | 62 612 227.421  | 24 604.328  | 24.638  | 62 636 856.388                                      |
| Molas (Lit)               | 62 602 356.570  | 34 452.784  | 44.926  | 62 636 854.279                                      |
| Ölands N. U. (Swe)        | 62 605 214.649  | 31 601.729  | 40.514  | 62 636 856.892                                      |
| Raahe (Fin)               | 62 616 878.076  | 19 941.164  | 34.653  | 62 636 853.893                                      |
| Ratan (Swe)               | 62 615 930.933  | 20 910.715  | 15.077  | 62 636 856.725                                      |
| Shepelevo (Rus)           | 62 609 584.472  | 27 231.134  | –   | –   |
| Spikarna (Swe)            | 62 613 440.933  | 23 395.355  | 18.895  | 62 636 855.183                                      |
| Stockholm (Swe)           | 62 608 441.995  | 28 295.629  | 116.888   | 62 636 854.512                                      |
| Swinoujscie (Pol)         | 62 599 126.796  | 37 663.497  | –   | –   |
| Ustka (Pol)               | 62 600 356.210  | 36 479.268  | 14.997  | 62 636 850.475                                      |
| Vaasa (Fin)               | 62 614 577.940  | 22 265.875  | 11.589  | 62 636 855.403                                      |
| Visby (Swe)               | 62 605 699.404  | 31 135.657  | 19.556  | 62 636 854.616                                      |
| Warnemünde (Ger)          | 62 599 530.263  | 37 213.313  | 111.089   | 62 636 854.665                                      |
|                           |   |   | Mean (m <sup>2</sup> /s <sup>2</sup> )  | 62 636 855.108                                      |
|                           |   |   | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                              | 0.3239  |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 28.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 2 (Kakkuri 1995), and gauge value of geoid potential  $W_0$  for the epoch of 1997.4

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 480.049   | 38 330.947   | 43.229   | 62 636 854.226   |
| Degerby (Fin)             | 62 609 699.381   | 27 125.102   | 28.248   | 62 636 852.731   |
| Furuögrund (Swe)          | 62 617 205.722   | 19 542.016   | 109.109  | 62 636 856.847   |
| Hamina (Fin)              | 62 610 584.135   | 26 255.426   | 16.663   | 62 636 856.224   |
| Hanko (Fin)               | 62 609 334.840   | 27 467.913   | 50.794   | 62 636 853.547   |
| Helgoland (Ger)           | 62 599 593.036   | 37 218.924   | —  | —  |
| Helsinki (Fin)            | 62 609 868.216   | 26 924.835   | 63.383   | 62 636 856.433   |
| kemi (Fin)                | 62 618 330.677   | 18 455.223   | 70.579   | 62 636 856.478   |
| Klagshamn (Swe)           | 62 602 014.413   | 34 818.792   | 20.602   | 62 636 853.807   |
| Klaipeda (Lit)            | 62 602 168.314   | 34 409.333   | —  | —  |
| Kronstadt (Rus)           | 62 609 616.854   | 27 194.595   | —  | —  |
| List/Sylt (Ger)           | 62 601 102.613   | 35 713.291   | 38.436   | 62 636 854.340   |
| Mäntyluoto (Fin)          | 62 612 227.421   | 24 604.328   | 25.159   | 62 636 856.908   |
| Molas (Lit)               | 62 602 356.570   | 34 452.784   | —  | —  |
| Ölands N. U. (Swe)        | 62 605 214.649   | 31 601.729   | 38.453   | 62 636 854.830   |
| Raahe (Fin)               | 62 616 878.076   | 19 941.164   | 35.429   | 62 636 854.669   |
| Ratan (Swe)               | 62 615 930.933   | 20 910.715   | 16.609   | 62 636 858.257   |
| Shepelevo (Rus)           | 62 609 584.472   | 27 231.134   | 42.045   | 62 636 857.650   |
| Spikarna (Swe)            | 62 613 440.933   | 23 395.355   | 23.471   | 62 636 859.760   |
| Stockholm (Swe)           | 62 608 441.995   | 28 295.629   | 118.086  | 62 636 855.710   |
| Swinoujscie (Pol)         | 62 599 126.796   | 37 663.497   | 65.568   | 62 636 855.860   |
| Ustka (Pol)               | 62 600 356.210   | 36 479.268   | —  | —  |
| Vaasa (Fin)               | 62 614 577.940   | 22 265.875   | 12.522   | 62 636 856.336   |
| Visby (Swe)               | 62 605 699.404   | 31 135.656   | 17.386   | 62 636 852.446   |
| Warnemünde (Ger)          | 62 599 530.263   | 37 213.313   | —  | —  |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 855.635   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.4350   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 29.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 3 (Kakkuri and Poutanen 1997), and gauge value of geoid potential  $W_0$  for the epoch of 1997.4.  $W_0$  at Ustka (Pol) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 480.049   | 38 330.947   | —  | —  |
| Degerby (Fin)             | 62 609 699.381   | 27 125.102   | 29.063   | 62 636 853.546   |
| Furuögrund (Swe)          | 62 617 205.722   | 19 542.016   | 108.558  | 62 636 856.296   |
| Hamina (Fin)              | 62 610 584.135   | 26 255.426   | 16.496   | 62 636 856.057   |
| Hanko (Fin)               | 62 609 334.840   | 27 467.913   | 51.177   | 62 636 853.930   |
| Helgoland (Ger)           | 62 599 593.036   | 37 218.924   | —  | —  |
| Helsinki (Fin)            | 62 609 868.216   | 26 924.835   | 63.471   | 62 636 856.522   |
| Kemi (Fin)                | 62 618 330.677   | 18 455.223   | 71.433   | 62 636 857.333   |
| Klagshamn (Swe)           | 62 602 014.413   | 34 818.792   | 16.853   | 62 636 850.058   |
| Klaipeda (Lit)            | 62 602 168.314   | 34 409.333   | —  | —  |
| Kronstadt (Rus)           | 62 609 616.854   | 27 194.595   | —  | —  |
| List/Sylt (Ger)           | 62 601 102.613   | 35 713.291   | —  | —  |
| Mäntyluoto (Fin)          | 62 612 227.421   | 24 604.328   | 25.296   | 62 636 857.046   |
| Molas (Lit)               | 62 602 356.570   | 34 452.784   | —  | —  |
| Ölands N. U. (Swe)        | 62 605 214.649   | 31 601.729   | 40.622   | 62 636 857.000   |
| Raahe (Fin)               | 62 616 878.076   | 19 941.164   | 35.832   | 62 636 855.072   |
| Ratan (Swe)               | 62 615 930.933   | 20 910.715   | 15.696   | 62 636 857.344   |
| Shepelevo (Rus)           | 62 609 584.472   | 27 231.134   | 42.546   | 62 636 858.151   |
| Spikarna (Swe)            | 62 613 440.933   | 23 395.355   | 20.182   | 62 636 856.470   |
| Stockholm (Swe)           | 62 608 441.995   | 28 295.629   | 118.950  | 62 636 856.574   |
| Swinoujscie (Pol)         | 62 599 126.796   | 37 663.497   | 62.025   | 62 636 852.317   |
| Ustka (Pol)               | 62 600 356.210   | 36 479.268   | 11.591   | 62 636 847.069   |
| Vaasa (Fin)               | 62 614 577.940   | 22 265.875   | 11.972   | 62 636 855.786   |
| Visby (Swe)               | 62 605 699.404   | 31 135.656   | 20.027   | 62 636 855.087   |
| Warnemünde (Ger)          | 62 599 530.263   | 37 213.313   | 108.831  | 62 636 852.407   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 855.389   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.5059   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 30.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 4 (Poutanen et al. 1999), and gauge value of geoid potential  $W_0$  for the epoch of 1997.4.  $W_0$  at Ustka (Pol) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|--|--|
| Borkum (Ger)              | 62 598 480.049   | 38 330.947   | 46.242   | 62 636 857.238   |
| Degerby (Fin)             | 62 609 699.381   | 27 125.102   | 28.611   | 62 636 853.095   |
| Furuögrund (Swe)          | 62 617 205.722   | 19 542.016   | 109.570  | 62 636 857.308   |
| Hamina (Fin)              | 62 610 584.135   | 26 255.426   | 17.724   | 62 636 857.284   |
| Hanko (Fin)               | 62 609 334.840   | 27 467.913   | 51.795   | 62 636 854.548   |
| Helgoland (Ger)           | 62 599 593.036   | 37 218.924   | 45.243   | 62 636 857.203   |
| Helsinki (Fin)            | 62 609 868.216   | 26 924.835   | 64.541   | 62 636 857.592   |
| Kemi (Fin)                | 62 618 330.677   | 18 455.223   | 72.612   | 62 636 858.512   |
| Klagshamn (Swe)           | 62 602 014.413   | 34 818.792   | 21.505   | 62 636 854.710   |
| Klaipeda (Lit)            | 62 602 168.314   | 34 409.333   | 278.131  | 62 636 855.779   |
| Kronstadt (Rus)           | 62 609 616.854   | 27 194.595   | 46.533   | 62 636 857.981   |
| List/Sylt (Ger)           | 62 601 102.613   | 35 713.291   | 41.077   | 62 636 856.980   |
| Mäntyluoto (Fin)          | 62 612 227.421   | 24 604.328   | 25.895   | 62 636 857.645   |
| Molas (Lit)               | 62 602 356.570   | 34 452.784   | 45.927   | 62 636 855.280   |
| Ölands N. U. (Swe)        | 62 605 214.649   | 31 601.729   | 41.633   | 62 636 858.011   |
| Raahe (Fin)               | 62 616 878.076   | 19 941.164   | 37.374   | 62 636 856.614   |
| Ratan (Swe)               | 62 615 930.933   | 20 910.715   | 16.982   | 62 636 858.631   |
| Shepelevo (Rus)           | 62 609 584.472   | 27 231.134   | 43.086   | 62 636 858.691   |
| Spikarna (Swe)            | 62 613 440.933   | 23 395.355   | 20.506   | 62 636 856.794   |
| Stockholm (Swe)           | 62 608 441.995   | 28 295.629   | 118.498  | 62 636 856.122   |
| Swinoujscie (Pol)         | 62 599 126.796   | 37 663.497   | 65.057   | 62 636 855.350   |
| Ustka (Pol)               | 62 600 356.210   | 36 479.268   | 14.869   | 62 636 850.347   |
| Vaasa (Fin)               | 62 614 577.940   | 22 265.875   | 13.759   | 62 636 857.574   |
| Visby (Swe)               | 62 605 699.404   | 31 135.656   | 20.233   | 62 636 855.293   |
| Warnemünde (Ger)          | 62 599 530.263   | 37 213.313   | 111.000  | 62 636 854.576   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )   | 62 636 856.617   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                 | 0.2980   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 31.** Gravitational potential  $U(\lambda, \phi, u)$ , centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 5 (Kakkuri 2000), and gauge value of geoid potential  $W_0$  for the epoch of 1997.4.  $W_0$  at Ustka (Pol) is an outlier, and not included in the mean

| Station name <sup>a</sup> | Gravitational potential<br>$U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential<br>$V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction<br>$\delta W(\lambda, \phi, u; H_0)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value<br>$W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|--|--|---|--|
| Borkum (Ger)              | 62 598 480.049   | 38 330.947   | 47.557  | 63 636 858.553   |
| Degerby (Fin)             | 62 609 699.381   | 27 125.102   | 30.192  | 62 636 854.675   |
| Furuögrund (Swe)          | 62 617 205.722   | 19 542.016   | 111.407   | 62 636 859.145   |
| Hamina (Fin)              | 62 610 584.135   | 26 255.426   | 19.334  | 62 636 858.895   |
| Hanko (Fin)               | 62 609 334.840   | 27 467.913   | 53.366  | 62 636 856.119   |
| Helgoland (Ger)           | 62 599 593.036   | 37 218.924   | 46.470  | 62 636 858.429   |
| Helsinki (Fin)            | 62 609 868.216   | 26 924.835   | 66.132  | 62 636 859.183   |
| Kemi (Fin)                | 62 618 330.677   | 18 455.223   | 74.380  | 62 636 860.280   |
| Klagshamn (Swe)           | 62 602 014.413   | 34 818.792   | 22.821  | 62 636 856.025   |
| Klaipeda (Lit)            | 62 602 168.314   | 34 409.333   | —   | —  |
| Kronstadt (Rus)           | 62 609 616.854   | 27 194.595   | —   | —  |
| List/Sylt (Ger)           | 62 601 102.613   | 35 713.291   | 42.362  | 62 636 858.266   |
| Mäntyluoto (Fin)          | 62 612 227.421   | 24 604.328   | —   | —  |
| Molas (Lit)               | 62 602 356.570   | 34 452.784   | 47.252  | 62 636 856.605   |
| Ölands N. U. (Swe)        | 62 605 214.649   | 31 601.729   | 43.057  | 62 636 859.434   |
| Raahe (Fin)               | 62 616 878.076   | 19 941.164   | 39.201  | 62 636 858.441   |
| Ratan (Swe)               | 62 615 930.933   | 20 910.715   | 18.780  | 62 636 860.428   |
| Shepelevo (Rus)           | 62 609 584.472   | 27 231.134   | —   | —  |
| Spikarna (Swe)            | 62 613 440.933   | 23 395.355   | 22.214  | 62 636 858.503   |
| Stockholm (Swe)           | 62 608 441.995   | 28 295.629   | 120.040   | 62 636 857.664   |
| Swinoujscie (Pol)         | 62 599 126.796   | 37 663.497   | 66.274  | 62 636 856.567   |
| Ustka (Pol)               | 62 600 356.210   | 36 479.268   | 16.126  | 62 636 851.604   |
| Vaasa (Fin)               | 62 614 577.940   | 22 265.875   | 15.704  | 62 636 859.518   |
| Visby (Swe)               | 62 605 699.404   | 31 135.656   | 21.676  | 62 636 856.736   |
| Warnemünde (Ger)          | 62 599 530.263   | 37 213.313   | 112.227   | 62 636 855.803   |
|                           |  |  | Mean (m <sup>2</sup> /s <sup>2</sup> )  | 62 636 857.963   |
|                           |  |  | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                                      | 0.3590   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 32.** Summary of  $w_0$  values computed from the GPS observations of the Baltic Sea Level project, third campaign, 1997.4, for cases 1–5

| Case | $w_0$ (m <sup>2</sup> /s <sup>2</sup> ) | Standard deviation ( $w_0$ ) (m <sup>2</sup> /s <sup>2</sup> ) |
|------|---|--|
| 1    | 62 636 855.108                          | 0.3239   |
| 2    | 62 636 855.635                          | 0.4350   |
| 3    | 62 636 855.389                          | 0.5059   |
| 4    | 62 636 856.617                          | 0.2980   |
| 5    | 62 636 857.963                          | 0.3590   |

However, for  $\dot{w}_0$  computation we need a different computation strategy. This strategy is motivated by the fact that in the relatively short time span from 1990.8 to 1997.4, the only variation in  $w_0$  which can be sensible is due to eustatic rise. Therefore, in order to remove the effects of any sources other than variations in sea surface, we again computed the  $w_0$  values for the epochs of 1990.8 and 1993.4 using the orthometric heights of cases 1–5, but the GPS coordinates of the Baltic Sea Level Project, third campaign, epoch 1997.4 only, which are the most accurate GPS observations made so far in the Baltic Sea.

Based on the new computation, we have produced the results shown in Tables 33–37. From a review of Tables 33–37 the following conclusions can be made.

1. The most consistent results correspond to case 1, where the heights are computed in their respective

national heights without application of regional geoid solutions.

2. Within the results of case 1, the most consistent result belong to the tide gauge stations of Germany.
3. From the  $\dot{w}_0$  computations at four stations [Borkum (Ger), Helgoland (Ger), List/Sylt (Ger), Ustka (Pol)] we have computed the following estimation for the time derivative of the geoid potential value  $\dot{w}_0$ :

$$\dot{w}_0 = -0.0099 \pm 0.00079 \text{ m}^2/\text{s}^2 \text{ per year} \quad (18)$$

## 8 Height datum difference of the countries around the Baltic Sea

The results we have presented in Table 27 provide us with a means to estimate the difference between the height datums of countries around the Baltic Sea. If now we rearrange Table 27 in such a way that the  $w_0$  values of the each country are kept together, we arrive at Table 38, which helps us to see the variation of  $w_0$  values from one country to the other. As is shown in Table 32 for each country we have computed a reference potential by averaging the two closest  $w_0$  at tide gauge stations of that country. Based on these reference potential values we computed the height datum difference between Finland, Germany, Lithuania and Sweden, as shown in Table 39 in gravity space in terms of potential difference, and in Table 40 in geometry space in terms of

**Table 33.** Results of  $\dot{w}_0$  computation based on the orthometric heights of case 1. The most consistent results correspond to tide gauge stations Borkum (Ger), Helgoland (Ger), List/Sylt (Ger) and Ustka (Pol)

| Station name <sup>a</sup> | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> ) per year<br>1993.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> ) per year<br>1997.4–1993.4 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> ) per year<br>1997.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> ) per year<br>mean |
|---------------------------|---|---|---|--|
| Borkum (Ger)              | –0.01153  | –0.00975  | –0.01045  | –0.01058   |
| Degerby (Fin)             | –0.05307  | 2.80830   | 1.68110   | 1.47870  |
| Furuögrund (Swe)          | 0.09076   | 0.08850   | 0.08939   | 0.08955  |
| Hamina (Fin)              | 0.01884   | 0.01725   | 0.01787   | 0.01799  |
| Hanko (Fin)               | 0.02653   | 8.22100   | 4.99290   | 4.41350  |
| Helgoland (Ger)           | –0.00769  | –0.01475  | –0.01197  | –0.01147   |
| Helsinki (Fin)            | 0.02269   | 0.93525   | 0.57576   | 0.51123  |
| Kemi (Fin)                | 0.07576   | 14.30700  | 8.70090   | 7.69460  |
| Klagshamn (Swe)           | 0   | –0.00250  | –0.00151  | –0.00133   |
| Klaipeda (Lit)            | –   | 0   | –   | –  |
| Kronstadt (Rus)           | –   | –   | –   | –  |
| List/Sylt (Ger)           | –0.00384  | –0.00975  | –0.00742  | –0.00700   |
| Mäntyluoto (Fin)          | 0.06423   | 0.06150   | 0.06257   | 0.06276  |
| Molas (Lit)               | –   | 0   | –   | –  |
| Ölands N. U. (Swe)        | 0.01538   | 0.01225   | 0.01348   | 0.01370  |
| Raahe (Fin)               | –0.76692  | 3.54100   | 1.84390   | 1.53930  |
| Ratan (Swe)               | 0.09038   | 0.08600   | 0.08772   | 0.08803  |
| Shepelevo (Rus)           | –   | –   | –   | –  |
| Spikarna (Swe)            | 0.07923   | 0.07600   | 0.07727   | 0.07750  |
| Stockholm (Swe)           | 0.03000   | –2.37600  | –1.42820  | –1.25810   |
| Swinoujście (Pol)         | –0.01115  | –   | –   | –  |
| Ustka (Pol)               | –0.01153  | –0.00975  | –0.01045  | –0.01058   |
| Vaasa (Fin)               | 0.07923   | 0.07600   | 0.07727   | 0.07750  |
| Visby (Swe)               | 0   | 0.01475   | 0.00893   | 0.00789  |
| Warnemünde (Ger)          | –   | –24.46700   | –   | –  |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 34.** Results of  $\dot{w}_0$  computation based on the orthometric heights of case 2. The results from one station to another are very different and therefore not acceptable

| Station name <sup>a</sup> | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1993.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1997.4–1993.4 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1997.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>mean |
|---------------------------|--|--|--|---|
| Borkum (Ger)              | −1.5477  | −0.0685  | −0.65121   | −0.7558   |
| Degerby (Fin)             | 0.12846  | 2.8695   | 1.7897   | 1.5959  |
| Furuögrund (Swe)          | 0.29846  | 0.0835   | 0.16818  | 0.18338   |
| Hamina (Fin)              | 0.075385   | −0.03175   | 0.010455   | 0.01803   |
| Hanko (Fin)               | 0.10962  | 8.221  | 5.0256   | 4.4521  |
| Helgoland (Ger)           | −  | −  | −  | −   |
| Helsinki (Fin)            | 0.19615  | 0.9475   | 0.65152  | 0.59839   |
| Kemi (Fin)                | 0.20385  | 14.317   | 8.7573   | 7.7594  |
| Klagshamn (Swe)           | 0.24923  | 0.09575  | 0.15621  | 0.16706   |
| Klaipeda (Lit)            | −  | −  | −  | −   |
| Kronstadt (Rus)           | −  | −  | −  | −   |
| List/Sylt (Ger)           | −0.24923   | −0.03425   | −0.11894   | −0.13414  |
| Mäntyluoto (Fin)          | 0.048846   | 0.07125  | 0.062424   | 0.06084   |
| Molas (Lit)               | −  | −  | −  | −   |
| Ölands N. U. (Swe)        | 0.21154  | 0.027  | 0.099697   | 0.11275   |
| Raahe (Fin)               | −1.0958  | 3.6047   | 1.753  | 1.4207  |
| Ratan (Swe)               | −0.36654   | 0.076  | −0.098333  | −0.12962  |
| Shepelevo (Rus)           | −  | 0.17425  | −  | −   |
| Spikarna (Swe)            | 0.17   | 0.0615   | 0.10424  | 0.11191   |
| Stockholm (Swe)           | 0.47231  | −2.4375  | −1.2912  | −1.0855   |
| Swinoujście (Pol)         | 0.31692  | 10.584   | 6.5397   | 5.8137  |
| Ustka (Pol)               | −  | −  | −  | −   |
| Vaasa (Fin)               | 0.21154  | 0.11775  | 0.1547   | 0.16133   |
| Visby (Swe)               | 0.42269  | −0.00975   | 0.16061  | 0.19118   |
| Warnemünde (Ger)          | −  | −  | −  | −   |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia**Table 35.** Results of  $\dot{w}_0$  computation based on the orthometric heights of case 3. The results from one station to another are very different and therefore not acceptable

| Station name <sup>a</sup> | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1993.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1997.4–1993.4 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1997.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>mean |
|---------------------------|--|--|--|---|
| Borkum (Ger)              | −  | −  | −  | −   |
| Degerby (Fin)             | 0.12846  | 2.8695   | 1.7897   | 1.5959  |
| Furuögrund (Swe)          | 0.29846  | 0.08325  | 0.16803  | 0.18325   |
| Hamina (Fin)              | 0.075769   | −0.032   | 0.010455   | 0.018075  |
| Hanko (Fin)               | 0.10962  | 8.221  | 5.0256   | 4.4521  |
| Helgoland (Ger)           | −  | −  | −  | −   |
| Helsinki (Fin)            | 0.19615  | 0.94775  | 0.65167  | 0.59852   |
| Kemi (Fin)                | 0.20423  | 14.317   | 8.7574   | 7.7596  |
| Klagshamn (Swe)           | 0.24923  | 0.09575  | 0.15621  | 0.16706   |
| Klaipeda (Lit)            | −  | −  | −  | −   |
| Kronstadt (Rus)           | −  | −  | −  | −   |
| List/Sylt (Ger)           | −  | −  | −  | −   |
| Mäntyluoto (Fin)          | 0.049231   | 0.07125  | 0.062576   | 0.061019  |
| Molas (Lit)               | −  | −  | −  | −   |
| Ölands N. U. (Swe)        | 0.21154  | 0.027  | 0.099697   | 0.11275   |
| Raahe (Fin)               | −1.0954  | 3.6047   | 1.7532   | 1.4208  |
| Ratan (Swe)               | −0.36654   | 0.07625  | −0.098182  | −0.12949  |
| Shepelevo (Rus)           | −  | 0.17425  | −  | −   |
| Spikarna (Swe)            | 0.17   | 0.0615   | 0.10424  | 0.11191   |
| Stockholm (Swe)           | 0.47231  | −2.4375  | −1.2912  | −1.0855   |
| Swinoujście (Pol)         | 0.31692  | 10.584   | 6.5397   | 5.8137  |
| Ustka (Pol)               | 0.22654  | −0.01475   | 0.080303   | 0.097364  |
| Vaasa (Fin)               | 0.21154  | 0.11775  | 0.1547   | 0.16133   |
| Visby (Swe)               | 0.42269  | −0.00975   | 0.16061  | 0.19118   |
| Warnemünde (Ger)          | 71.218   | −24.676  | 13.101   | 19.881  |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 36.** Results of  $\dot{w}_0$  computation based on the orthometric heights of case 4. The results from one station to another are very different and therefore not acceptable

| Station name <sup>a</sup> | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1993.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1997.4–1993.4 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1997.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>mean |
|---------------------------|--|--|--|---|
| Borkum (Ger)              | −1.5477  | −0.06875   | −0.65136   | −0.75594  |
| Degerby (Fin)             | 0.12846  | 2.8695   | 1.7897   | 1.5959  |
| Furuögrund (Swe)          | 0.29846  | 0.0835   | 0.16818  | 0.18338   |
| Hamina (Fin)              | 0.075385   | −0.032   | 0.010303   | 0.017896  |
| Hanko (Fin)               | 0.10962  | 8.2208   | 5.0255   | 4.4519  |
| Helgoland (Ger)           | −0.0038462   | 0.118  | 0.07   | 0.061385  |
| Helsinki (Fin)            | 0.19654  | 0.9475   | 0.65167  | 0.59857   |
| Kemi (Fin)                | 0.20385  | 14.317   | 8.7574   | 7.7595  |
| Klagshamn (Swe)           | 0.24923  | 0.09575  | 0.15621  | 0.16706   |
| Klaipeda (Lit)            | —  | 0.032  | —  | —   |
| Kronstadt (Rus)           | —  | —  | —  | —   |
| List/Sylt (Ger)           | −0.24885   | −0.0345  | −0.11894   | −0.1341   |
| Mäntyluoto (Fin)          | 0.049231   | 0.07125  | 0.062576   | 0.061019  |
| Molas (Lit)               | —  | 0.03675  | —  | —   |
| Ölands N. U. (Swe)        | 0.21154  | 0.027  | 0.099697   | 0.11275   |
| Raahe (Fin)               | −1.0954  | 3.6047   | 1.7532   | 1.4208  |
| Ratan (Swe)               | −0.36654   | 0.07625  | −0.098182  | −0.12949  |
| Shepelevo (Rus)           | —  | 0.17425  | —  | —   |
| Spikarna (Swe)            | 0.16962  | 0.0615   | 0.10409  | 0.11174   |
| Stockholm (Swe)           | 0.47192  | −2.4375  | −1.2914  | −1.0856   |
| Swinoujście (Pol)         | 0.31731  | 10.584   | 6.5398   | 5.8139  |
| Ustka (Pol)               | 0.22654  | −0.01475   | 0.080303   | 0.097364  |
| Vaasa (Fin)               | 0.21154  | 0.118  | 0.15485  | 0.16146   |
| Visby (Swe)               | 0.42269  | −0.00975   | 0.16061  | 0.19118   |
| Warnemünde (Ger)          | 71.218   | −24.676  | 13.101   | 19.881  |

<sup>a</sup>Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia**Table 37.** Results of  $\dot{w}_0$  computation based on the orthometric heights of case 5. The results from one station to another are very different and therefore not acceptable

| Station name <sup>a</sup> | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1993.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1997.4–1993.4 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>1997.4–1990.8 | $\dot{w}_0$ (m <sup>2</sup> /s <sup>2</sup> per year)<br>mean |
|---------------------------|--|--|--|---|
| Borkum (Ger)              | −1.5477  | −0.06875   | −0.65136   | −0.75594  |
| Degerby (Fin)             | 0.12808  | 2.8695   | 1.7895   | 1.5957  |
| Furuögrund (Swe)          | 0.29846  | 0.0835   | 0.16818  | 0.18338   |
| Hamina (Fin)              | 0.075385   | −0.03175   | 0.010455   | 0.01803   |
| Hanko (Fin)               | 0.10962  | 8.2208   | 5.0255   | 4.4519  |
| Helgoland (Ger)           | −0.0038462   | 0.11775  | 0.069848   | 0.061251  |
| Helsinki (Fin)            | 0.19615  | 0.94775  | 0.65167  | 0.59852   |
| Kemi (Fin)                | 0.20423  | 14.317   | 8.7574   | 7.7596  |
| Klagshamn (Swe)           | 0.24923  | 0.0955   | 0.15606  | 0.16693   |
| Klaipeda (Lit)            | —  | —  | —  | —   |
| Kronstadt (Rus)           | —  | —  | —  | —   |
| List/Sylt (Ger)           | −0.24923   | −0.03425   | −0.11894   | −0.13414  |
| Mäntyluoto (Fin)          | —  | —  | —  | —   |
| Molas (Lit)               | —  | 0.03675  | —  | —   |
| Ölands N. U. (Swe)        | 0.21115  | 0.027  | 0.099545   | 0.11257   |
| Raahe (Fin)               | −1.0954  | 3.6047   | 1.7532   | 1.4208  |
| Ratan (Swe)               | −0.36615   | 0.076  | −0.098182  | −0.12945  |
| Shepelevo (Rus)           | —  | —  | —  | —   |
| Spikarna (Swe)            | 0.17   | 0.0615   | 0.10424  | 0.11191   |
| Stockholm (Swe)           | 0.47231  | −2.4375  | −1.2912  | −1.0855   |
| Swinoujście (Pol)         | 0.31731  | 10.584   | 6.5398   | 5.8139  |
| Ustka (Pol)               | 0.22654  | −0.01475   | 0.080303   | 0.097364  |
| Vaasa (Fin)               | 0.21154  | 0.11775  | 0.1547   | 0.16133   |
| Visby (Swe)               | 0.42269  | −0.00975   | 0.16061  | 0.19118   |
| Warnemünde (Ger)          | 71.218   | −24.676  | 13.101   | 19.881  |

<sup>a</sup>Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 38.** Gravitational potential  $U(\lambda, \phi, u)$ ; centrifugal potential  $V(\lambda, \phi, u)$ , free-air gravity potential  $\Delta W(\lambda, \phi, u; H_0)$ , computed based on orthometric heights of case 1 (Poutanen et al. 1999), and gauge value of geoid potential  $W_0$  for the epoch of 1997.4. **Bold** stations are those used for computing the reference potentials

| Station name <sup>a</sup> | Gravitational potential $U(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Centrifugal potential $V(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Free-air reduction $\delta W(\lambda, \phi, u)$ (m <sup>2</sup> /s <sup>2</sup> ) | Gauge value $W_0$ (m <sup>2</sup> /s <sup>2</sup> ) |
|---------------------------|---|---|---|---|
| Borkum (Ger)              | 62 598 480.049  | 38 330.947  | 44.888  | 62 636 855.884                                      |
| <b>Helgoland (Ger)</b>    | <b>62 599 593.036</b>   | <b>37 218.924</b>   | <b>44.468</b>   | <b>62 636 856.427</b>                               |
| <b>List/Sylt (Ger)</b>    | <b>62 601 102.613</b>   | <b>35 713.291</b>   | <b>40.782</b>   | <b>62 636 856.686</b>                               |
| Warnemünde (Ger)          | 62 599 530.263  | 37 213.313  | 111.089   | 62 636 854.665                                      |
|                           |   |   | Mean (m <sup>2</sup> /s <sup>2</sup> )  | 62 636 856.557                                      |
|                           |   |   | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                              | 0.129   |
| Degerby (Fin)             | 62 609 699.381  | 27 125.102  | 27.738  | 62 636 852.221                                      |
| <b>Hamina (Fin)</b>       | <b>62 610 584.135</b>   | <b>26 255.426</b>   | <b>16.015</b>   | <b>62 636 855.576</b>                               |
| Hanko (Fin)               | 62 609 334.840  | 27 467.913  | 50.254  | 62 636 853.007                                      |
| Helsinki (Fin)            | 62 609 868.216  | 26 924.835  | 63.039  | 62 636 856.090                                      |
| <b>Kemi (Fin)</b>         | <b>62 618 330.677</b>   | <b>18 455.223</b>   | <b>69.665</b>   | <b>62 636 855.565</b>                               |
| Mäntyluoto (Fin)          | 62 612 227.421  | 24 604.328  | 24.638  | 62 636 856.388                                      |
| Raahe (Fin)               | 62 616 878.076  | 19 941.164  | 34.653  | 62 636 853.893                                      |
| <b>Vaasa (Fin)</b>        | <b>62 614 577.940</b>   | <b>22 265.875</b>   | <b>11.589</b>   | <b>62 636 855.403</b>                               |
|                           |   |   | Mean (m <sup>2</sup> /s <sup>2</sup> )  | 62 636 855.515                                      |
|                           |   |   | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                              | 0.056   |
| Furuögrund (Swe)          | 62 617 205.722  | 19 542.016  | 107.733   | 62 636 855.511                                      |
| Klagshamn (Swe)           | 62 602 014.413  | 34 818.792  | 20.004  | 62 636 853.208                                      |
| <b>Ölands N.U. (Swe)</b>  | <b>62 605 214.649</b>   | <b>31 601.729</b>   | <b>40.514</b>   | <b>62 636 856.892</b>                               |
| <b>Ratan (Swe)</b>        | <b>62 615 930.933</b>   | <b>20 910.715</b>   | <b>15.077</b>   | <b>62 636 856.725</b>                               |
| Spikarna (Swe)            | 62 613 440.933  | 23 395.355  | 18.895  | 62 636 855.183                                      |
| Stockholm (Swe)           | 62 608 441.995  | 28 295.629  | 116.888   | 62 636 854.512                                      |
| Visby (Swe)               | 62 605 699.404  | 31 135.656  | 19.556  | 62 636 854.616                                      |
|                           |   |   | Mean (m <sup>2</sup> /s <sup>2</sup> )  | 62 636 856.512                                      |
|                           |   |   | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                              | 0.083   |
| <b>Klaipeda (Lit)</b>     | <b>62 602 168.314</b>   | <b>34 409.333</b>   | <b>276.885</b>  | <b>62 636 854.532</b>                               |
| <b>Molas (Lit)</b>        | <b>62 602 356.570</b>   | <b>34 452.784</b>   | <b>44.926</b>   | <b>62 636 854.279</b>                               |
|                           |   |   | Mean (m <sup>2</sup> /s <sup>2</sup> )  | 62 636 854.406                                      |
|                           |   |   | Standard deviation (m <sup>2</sup> /s <sup>2</sup> )                              | 0.126   |
| Kronstadt (Rus)           | 62 609 616.854  | 27 194.595  | –   | –   |
| Shepelevo (Rus)           | 62 609 584.472  | 27 231.134  | –   | –   |
| Swinoujście (Pol)         | 62 599 126.796  | 37 663.497  | –   | –   |
| Ustka (Pol)               | 62 600 356.210  | 36 479.268  | 14.997  | 62 636 850.475                                      |

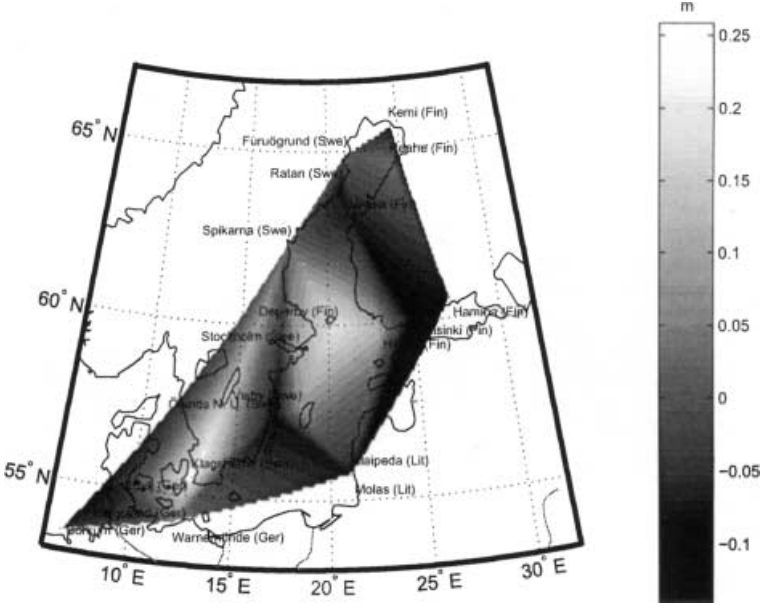
<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 39.** Datum difference between the countries around the Baltic Sea, in potential units

|           | Finland  | Germany  | Lithuania   | Sweden   |
|-----------|--|--|---|--|
| Finland   | 0  | (−1.042 ± 0.070) (m <sup>2</sup> /s <sup>2</sup> ) | (1.109 ± 0.069) (m <sup>2</sup> /s <sup>2</sup> ) | (−1.294 ± 0.050) (m <sup>2</sup> /s <sup>2</sup> ) |
| Germany   | (1.042 ± 0.070) (m <sup>2</sup> /s <sup>2</sup> )  | 0  | (2.151 ± 0.090) (m <sup>2</sup> /s <sup>2</sup> ) | (−0.252 ± 0.077) (m <sup>2</sup> /s <sup>2</sup> ) |
| Lithuania | (−1.109 ± 0.069) (m <sup>2</sup> /s <sup>2</sup> ) | (−2.151 ± 0.090) (m <sup>2</sup> /s <sup>2</sup> ) | 0   | (−2.403 ± 0.075) (m <sup>2</sup> /s <sup>2</sup> ) |
| Sweden    | (1.294 ± 0.050) (m <sup>2</sup> /s <sup>2</sup> )  | (0.252 ± 0.077) (m <sup>2</sup> /s <sup>2</sup> )  | (2.403 ± 0.075) (m <sup>2</sup> /s <sup>2</sup> ) | 0  |

**Table 40.** Datum difference between the countries around the Baltic Sea, in metres, based on mean value of vertical gradient of gravity −9.81802523 m/s<sup>2</sup> in the Baltic Sea area

|           | Finland              | Germany              | Lithuania           | Sweden               |
|-----------|----------------------|----------------------|---------------------|----------------------|
| Finland   | 0                    | (−0.106 ± 0.007) (m) | (0.113 ± 0.007) (m) | (−0.132 ± 0.005) (m) |
| Germany   | (0.106 ± 0.007) (m)  | 0                    | (0.219 ± 0.009) (m) | (−0.026 ± 0.008) (m) |
| Lithuania | (−0.113 ± 0.007) (m) | (−0.219 ± 0.009) (m) | 0                   | (−0.245 ± 0.008) (m) |
| Sweden    | (0.132 ± 0.005) (m)  | (0.026 ± 0.008) (m)  | (0.245 ± 0.008) (m) | 0                    |



**Fig. 2.** Computed SST map of the Baltic Sea in German height datum. Equidistant conic map projection with two standard parallels: 55° and 65°N, and reference ellipsoid of WGD2000

**Table 41.** Transformation relation of potential difference into height difference

Decomposition of actual geoid potential value  $w_0$  into the apparent geoid potential value at the tide gauges  $W_{0i}$ , and the disturbing part  $\delta W$

$$w_0 = W_{0i} + \delta W = W_{0i} + \nabla_N W \cdot (u_0 - u)$$

$$= W_{0i} + \frac{1}{\sqrt{g_{uu}}} \frac{\partial W}{\partial u} \cdot \sqrt{g_{uu}}(u_0 - u) = W_{0i} + \frac{1}{\sqrt{g_{uu}}} \frac{\partial W}{\partial u} \cdot \Delta u^{(1)} \quad (19)$$

The physical height difference (i.e. SST)

$$\Delta u^{(1)} = \frac{(w_0 - W_{0i}) \sqrt{g_{uu}}}{\frac{\partial W}{\partial u}} \quad (20)$$

subject to

$$\frac{\partial W}{\partial u} = \sum_{n=0}^{360} \sum_{m=-n}^{+n} u_{nm} \frac{1}{Q_{n|m}^* \left( \frac{b}{a} \right)} \frac{\partial Q_{n|m}^* \left( \frac{a}{b} \right)}{\partial u} e_{nm}(\lambda, \phi)$$

$$+ \omega^2 u \sqrt{u^2 + e^2} \cos^2 \phi \quad (21)$$

height difference. To convert the potential differences into height differences (i.e. to transformation from gravity space to geometry space) we use the mean value of vertical gradient of gravity  $-9.81802523 \text{ m/s}^2$  in the Baltic Sea area, based on an ellipsoidal harmonic model of degree/order 360/360. The details of the computation of the vertical gradient of gravity we discussed in the next section.

## 9 Sea surface topography map of the Baltic Sea

Having access to the MSL information at various tide gauge stations around the Baltic Sea, we have the opportunity to derive the sea surface topography (SST) of the Baltic Sea. Here we use the GPS observations of the Baltic Sea Level Project, third campaign, which are the most accurate ones, and the orthometric heights of case 1, which are the directly observed heights above MSL in the height datum of various countries (see Table

27). Of course, now that we have estimated the datum difference between the countries around the Baltic sea we can unify the datum of the orthometric heights of case 1.

The difference between the  $w_0$  values presented in Table 27 and the average value  $w_0 = 62636855.75 \pm 0.21 \text{ m}^2/\text{s}^2$  provides us with the apparent SST, i.e. the deviation of the sea surface from the geoid at various tide gauge stations around the Baltic Sea, plus the height datum difference. These potential deviations can be converted into metric units according to the transformation relation outlined in Table 41.

As shown in Table 41, the geoid potential value  $w_0$  can be decomposed into the computed (or apparent) geoid potential value at the various tide gauges  $W_{0i}$  (index  $i$  runs from one to the total number of tide gauge stations), and the disturbing part  $\delta W$ . The disturbing part is due to the SST ( $u_0 - u$ ). The SST ( $u_0 - u$ ) is in terms of Jacobi ellipsoidal coordinates, and as such is not a physical component. However,  $\Delta u^{(1)} = \sqrt{g_{uu}}(u_0 - u)$ , derived by using the directional derivative operator [see Eqs. (19) and (20)], is the required physical height. For the derivative of the potential with respect to  $u$  appearing in Eq. (20), we use the high-resolution model of Eq. (21).

Table 42 presents the computed SST,  $\Delta u^{(1)}$ , of the Baltic Sea at various tide gauges stations. We have also corrected the SST of tide gauge stations for the difference between the national height datums, to produce the last column of Table 42, which shows the SST of the Baltic Sea Level tide gauges in the German height datum. Note that the SST could be more stable with respect to the German height datum than the height datums of the countries in the northern part of the Baltic Sea, for example, particularly due to the runoff of rivers in the north in the spring period. This is why we have computed the SST of the Baltic Sea in the height datum of Germany. A contour map plot of computed SST for the Baltic Sea is shown in Fig. 2.

Table 43 shows a comparison of the computed SST for the Baltic Sea as explained above and the one computed by Kakkuri and Poutanen (1997). The

difference between the two SST solutions is  $0.011 \pm 0.054$  m.

**Table 42.** Vertical gradient of potential, difference between apparent  $W_{0i}$  based on height of the stations in their respective national height systems and average value of geoid potential  $w_0$ , apparent SST at the tide gauge stations of Baltic Sea Level project in different height datums, corrections to German height datum, and corrected SST in German height datum

| Station name <sup>a</sup> | $\frac{1}{\sqrt{g_{uu}}} \frac{\partial W}{\partial u}$ (m/s <sup>2</sup> ) | $(w_0 - W_0)$ (m <sup>2</sup> /s <sup>2</sup> ) | $\delta u^{(1)} = \frac{(w_0 - W_0)}{\frac{1}{\sqrt{g_{uu}}} \frac{\partial W}{\partial u}}$ (m) | Correction to German height datum (m) | Corrected SST (m) |
|---------------------------|---|---|--|---------------------------------------|-------------------|
| Borkum (Ger)              | -9.81363626   | -0.134  | 0.0137   | 0                                     | -0.014            |
| Degerby (Fin)             | -9.81862941   | 3.529   | -0.3594  | 0.106                                 | 0.253             |
| Furuögrund (Swe)          | -9.82251837   | 0.239   | -0.0243  | -0.026                                | 0.050             |
| Hamina (Fin)              | -9.81923277   | 0.174   | -0.0177  | 0.106                                 | -0.088            |
| Hanko (Fin)               | -9.81899359   | 2.743   | -0.2794  | 0.106                                 | 0.173             |
| Helgoland (Ger)           | -9.81407564   | -0.677  | 0.0690   | 0                                     | -0.069            |
| Helsinki (Fin)            | -9.81916005   | -0.340  | 0.0346   | 0.106                                 | -0.141            |
| Kemi (Fin)                | -9.82306382   | 0.185   | -0.0188  | 0.106                                 | -0.087            |
| Klagshamn (Swe)           | -9.81537974   | 2.542   | -0.2590  | -0.026                                | 0.285             |
| Klaipeda (Lit)            | -9.81547717   | 1.218   | -0.1241  | 0.219                                 | -0.095            |
| Kronstadt (Rus)           | -9.81913536   | -   | -  | -                                     | -                 |
| List/Sylt (Ger)           | -9.81521803   | -0.936  | 0.0954   | 0                                     | -0.095            |
| Mäntyluoto (Fin)          | -9.82002342   | -0.638  | 0.0650   | 0.106                                 | -0.171            |
| Molas (Lit)               | -9.81549953   | 1.471   | -0.1499  | 0.219                                 | -0.069            |
| Ölands N. U. (Swe)        | -9.81683093   | -1.142  | 0.1163   | -0.026                                | -0.090            |
| Raahe (Fin)               | -9.82228525   | 1.857   | -0.1891  | 0.106                                 | 0.083             |
| Ratan (Swe)               | -9.82207194   | -0.975  | 0.0993   | -0.026                                | -0.073            |
| Shepelevo (Rus)           | -9.81897667   | -   | -  | -                                     | -                 |
| Spikarna (Swe)            | -9.82068805   | 0.567   | -0.0577  | -0.026                                | 0.084             |
| Stockholm (Swe)           | -9.81840031   | 1.238   | -0.1261  | -0.026                                | 0.152             |
| Swinoujscie (Pol)         | -9.81406739   | -   | -  | -                                     | -                 |
| Ustka (Pol)               | -9.81472950   | 5.275   | -0.5375  | -                                     | -                 |
| Vaasa (Fin)               | -9.82096810   | 0.347   | -0.0353  | 0.106                                 | -0.071            |
| Visby (Swe)               | -9.81722771   | 1.134   | -0.1155  | -0.026                                | 0.142             |
| Warnemünde (Ger)          | -9.81434182   | 1.085   | -0.1106  | 0                                     | 0.111             |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 43.** Comparison of authors' SST with the SST computed by Kakkuri and Poutanen (1997) for the Baltic Sea

| Station name <sup>a</sup> | SST (m) | SST (Kakkuri and Poutanen 1997) (m) | Difference between SST's (m) |
|---------------------------|---------|-------------------------------------|------------------------------|
| Borkum (Ger)              | -0.014  | -                                   | -                            |
| Degerby (Fin)             | 0.253   | 0.095                               | 0.158                        |
| Furuögrund (Swe)          | 0.050   | 0.054                               | -0.004                       |
| Hamina (Fin)              | -0.088  | 0.052                               | -0.140                       |
| Hanko (Fin)               | 0.173   | 0.079                               | 0.094                        |
| Helgoland (Ger)           | -0.069  | -                                   | -                            |
| Helsinki (Fin)            | -0.141  | 0.024                               | -0.165                       |
| Kemi (Fin)                | -0.087  | 0.146                               | -0.233                       |
| Klagshamn (Swe)           | 0.285   | -0.363                              | 0.648                        |
| Klaipeda (Lit)            | -0.095  | -                                   | -                            |
| Kronstadt (Rus)           | -       | -                                   | -                            |
| List/Sylt (Ger)           | -0.095  | -                                   | -                            |
| Mäntyluoto (Fin)          | -0.171  | 0.044                               | -0.215                       |
| Molas (Lit)               | -0.069  | -                                   | -                            |
| Ölands N. U. (Swe)        | -0.090  | -0.03                               | -0.060                       |
| Raahe (Fin)               | 0.083   | 0.067                               | 0.016                        |
| Ratan (Swe)               | -0.073  | 0.041                               | -0.114                       |
| Shepelevo (Rus)           | -       | 0.260                               | -                            |
| Spikarna (Swe)            | 0.084   | 0.116                               | -0.032                       |
| Stockholm (Swe)           | 0.152   | 0.199                               | -0.047                       |
| Swinoujscie (Pol)         | -       | -0.300                              | -                            |
| Ustka (Pol)               | -       | -0.344                              | -                            |
| Vaasa (Fin)               | -0.071  | -0.01                               | -0.061                       |
| Visby (Swe)               | 0.142   | 0.050                               | 0.092                        |
| Warnemünde (Ger)          | 0.111   | -0.143                              | 0.254                        |

<sup>a</sup> Ger: Germany; Fin: Finland; Swe: Sweden; Lit: Lithuania; Pol: Poland; Rus: Russia

**Table 44.** Comparison of potential value of geoid  $w_0$  computed by various authors

| Author                       | $w_0$ value ( $\text{m}^2/\text{s}^2$ ) | Data source computation method  |
|------------------------------|---|---|
| Nesvorný and Šíma (1994)     | $62\,636\,857.5 \pm 1.0$                | Satellite altimetry data  |
| Burša et al. (1997a)         | $62\,636\,855.72 \pm 0.5$               | Satellite altimetry data, gauge station values  |
| Burša et al. (1997b)         | $62\,636\,855.80 \pm 0.5$               | Satellite altimetry data, gauge station values  |
| Grafarend and Ardalan (1997) | $62\,636\,855.8 \pm 3.6$                | Ellipsoidal harmonic expansion, and tide gauge information of Baltic Sea, GPS observations of Baltic Sea Level projects second campaign                   |
| Burša et al. (1998)          | $62\,636\,855.611 \pm 0.5$              | TOPEX/POSEIDON altimeter data   |
| Burša et al. (2000)          | $62\,636\,856.0 \pm 0.5$                | TOPEX/POSEIDON altimeter data   |
| Our results                  | $62\,636\,855.75 \pm 0.21$              | Ellipsoidal harmonic expansion, and tide gauge information of Baltic Sea, GPS observations of Baltic Sea Level Project, first, second and third campaigns |

## 10 Conclusions

The results we have obtained in the various sections can be summarised as follows.

1. Our best estimate for the  $w_0$  value based on the GPS observations of the Baltic Sea Level project, first, second and third campaigns, and tide gauge observations is

$$w_0 = 62\,636\,855.75 \pm 0.21 \text{ m}^2/\text{s}^2$$

2. Our best estimate for the  $\dot{w}_0$  value based on the GPS observations of the Baltic Sea Level project, first, second and third campaigns, and tide gauge observations is

$$\dot{w}_0 = -0.0099 \pm 0.00079 \text{ m}^2/\text{s}^2 \text{ per year}$$

or

$$\dot{w}_0/\bar{\gamma} = 1.0 \text{ mm/year}$$

3. Amongst the different geoid solutions proposed for the Baltic Sea, the one introduced in Poutanen et al. (1999) is the most accurate one.
4. The geoid proposed by Kakkuri (2000) has a shift of approximately 0.244 m.
5. Table 44 shows a comparison with already published  $w_0$  values.

Finally, based on the results obtained we can conclude that our method is quite successful for the computation of the fundamental geodetic parameter  $w_0$ , the potential value of the Gauss–Listing geoid, as well as its time derivative  $\dot{w}_0$ . In addition, the proposed methodology can also be quite helpful in (1) unification of national height datums, (2) computation of high-resolution SST maps, and (3) accuracy estimation of the geoid solutions tailored to sea areas.

*Acknowledgement.* A. Ardalan acknowledges gratefully the financial support which was provided to him by the University of Tehran under grant number 621/3/600.

## Appendix A

### Spheroidal coordinates

It was revealed by the great early-18th-century expeditions that the Earth is *not* geometrically a sphere, but

nearly an oblate ellipsoid-of-revolution  $\mathbb{E}_{a,a,b}^2$ . [See e.g. Kakkuri et al. (1986); Smith (1986); Tobé (1986) for a historical review of the progress on the determination of the shape of the Earth]. Therefore, representation of the gravity field of the Earth in terms of ellipsoidal harmonics is more accurate and even more convenient than, for example, spherical coordinates and spherical harmonics.

Here we will briefly review the main features of ellipsoidal coordinates and ellipsoidal harmonic expansion and invite interested readers to consult Thong and Grafarend (1989).

**Definition A1.** (*spheroidal coordinates*  $\{\lambda, \phi, u\}$ ). In terms of ellipsoidal coordinates  $\{\lambda, \phi, u\}$ , a point in space can be located as the intersection point of the following families of surfaces.

1. The family of confocal, oblate spheroids:

$$\mathbb{E}_{\sqrt{u^2+\epsilon^2},u}^2 := \left\{ \mathbf{x} \in \mathbb{R}^3 \left| \frac{x^2+y^2}{u^2+\epsilon^2} + \frac{z^2}{u^2} = 1, \right. \right. \\ \left. \left. u \in (0, +\infty), \epsilon^2 := a^2 - b^2 \right\} \quad (\text{A1})$$

2. The family of confocal half hyperboloids:

**Table A1.** Forward and backward transformation of Cartesian coordinates  $\{x, y, z\}$  into ellipsoidal coordinates  $\{\lambda, \phi, u\}$ 

(i) Forward transformation,  $\{\lambda, \phi, u\} \mapsto \{x, y, z\}$

$$x = \sqrt{u^2 + \epsilon^2} \cos \phi \cos \lambda$$

$$y = \sqrt{u^2 + \epsilon^2} \cos \phi \sin \lambda$$

$$z = u \sin \phi$$

(A4)

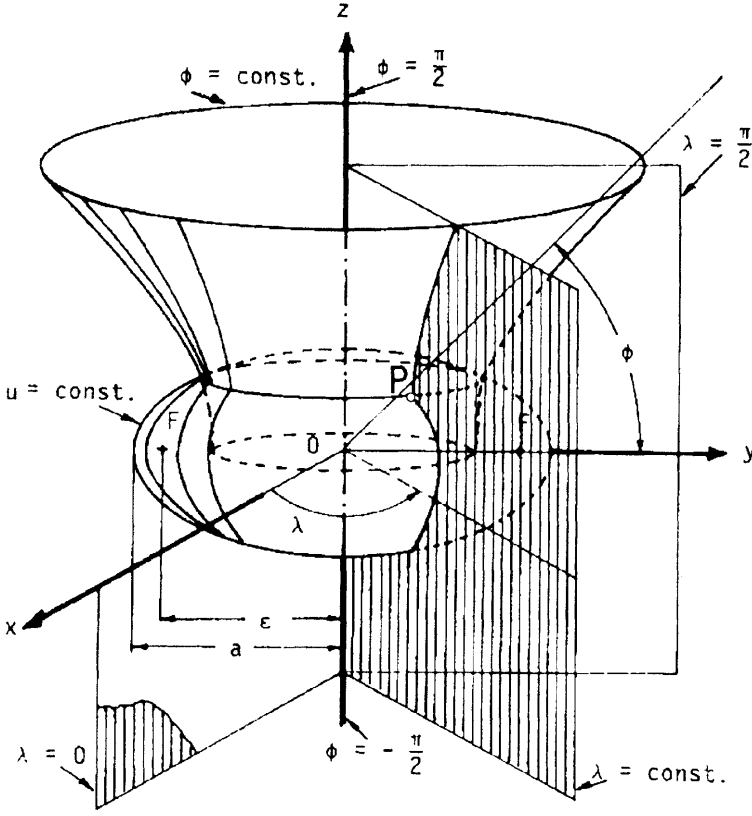
(ii) Backward transformation,  $\{x, y, z\} \mapsto \{\lambda, \phi, u\}$

$$\lambda = \begin{cases} \arctan \frac{y}{x} & \text{for } x > 0 \text{ and } y \geq 0 \\ \arctan \frac{y}{x} + \pi & \text{for } x < 0 \text{ and } y \neq 0 \\ \arctan \frac{y}{x} + 2\pi & \text{for } x > 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{for } x = 0 \text{ and } y > 0 \\ \frac{3\pi}{2} & \text{for } x = 0 \text{ and } y < 0 \end{cases} \quad (\text{A5})$$

$$\phi = (\text{sgn } z) \arcsin \left\{ \frac{1}{2\epsilon^2} \left[ \epsilon^2 - (x^2 + y^2 + z^2) \right. \right.$$

$$\left. \left. + \sqrt{(x^2 + y^2 + z^2 - \epsilon^2)^2 + 4\epsilon^2 z^2} \right] \right\}^{1/2} \quad (\text{A6})$$

$$u = \left\{ \frac{1}{2} \left[ x^2 + y^2 + z^2 - \epsilon^2 + \sqrt{(x^2 + y^2 + z^2 - \epsilon^2)^2 + 4\epsilon^2 z^2} \right] \right\}^{1/2} \quad (\text{A7})$$



**Fig. A1.** Spheroidal coordinates  $\{\lambda, \phi, u\}$ . The coordinate surfaces are of the types: (1) spheroids ( $u = \text{const.}$ ), (2) half hyperboloids of one sheet ( $\phi = \text{const.}$ ), and (3) half planes ( $\lambda = \text{const.}$ )

$$\mathbb{H}_{\epsilon \cos \phi, \epsilon \sin \phi}^2 := \left\{ \mathbf{x} \in \mathbb{R}^3 \left| \frac{x^2 + y^2}{\epsilon^2 \cos^2 \phi} - \frac{z^2}{\epsilon^2 \sin^2 \phi} = 1, \right. \right. \\ \left. \left. \phi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], \phi \neq 0 \right\} \quad (\text{A2})$$

$$X_\lambda = D_\lambda X = -\sqrt{u^2 + \epsilon^2} \cos \phi \sin \lambda$$

$$Y_\lambda = D_\lambda Y = \sqrt{u^2 + \epsilon^2} \cos \phi \cos \lambda$$

$$Z_\lambda = D_\lambda Z = 0$$

$$X_\phi = D_\phi X = -\sqrt{u^2 + \epsilon^2} \sin \phi \cos \lambda$$

$$Y_\phi = D_\phi Y = -\sqrt{u^2 + \epsilon^2} \sin \phi \sin \lambda$$

$$Z_\phi = D_\phi Z = u \cos \phi$$

$$X_u = D_u X = \frac{u}{\sqrt{u^2 + \epsilon^2}} \cos \phi \cos \lambda$$

$$Y_u = D_u Y = \frac{u}{\sqrt{u^2 + \epsilon^2}} \cos \phi \sin \lambda$$

$$Z_u = D_u Z = \sin \phi$$

2. The metric tensor

$$dS^2 = [d\lambda, d\phi, du] J^* J \begin{bmatrix} d\lambda \\ d\phi \\ du \end{bmatrix} \quad (\text{A9})$$

3. The family of half planes:

$$\mathbb{P}_{\cos \lambda, \sin \lambda}^2 := \{ \mathbf{x} \in \mathbb{R}^3 | y = x \tan \lambda, \lambda \in [0, 2\pi] \} \quad (\text{A3})$$

As shown in Fig. A1, longitude  $\lambda$  gives *orientation* to the half planes, latitude  $\phi$  is the *inclination* of asymptotes of the confocal half hyperboloids, and the elliptic coordinate  $u$  is the *semi-minor axis* of confocal oblate spheroids (confocal, oblate ellipsoids of revolution).

**Definition A2.** Basic geometry of ellipsoidal coordinates  $\{\lambda, \phi, u\}$

1. Jacobi matrix of the forward transformation  $\{\lambda, \phi, u\} \mapsto \{x, y, z\}$

From Eq. (A4) the Jacobi matrix  $J$  of the transformation from ellipsoidal coordinates  $\{\lambda, \phi, u\}$  into Cartesian coordinates  $\{x, y, z\}$  can be obtained as follows.

$$J := \begin{bmatrix} X_\lambda & X_\phi & X_u \\ Y_\lambda & Y_\phi & Y_u \\ Z_\lambda & Z_\phi & Z_u \end{bmatrix} \quad (\text{A8})$$

The partial derivatives used in Eq. (A8) are defined as

$$G := J^* J \begin{bmatrix} (u^2 + \epsilon^2) \cos^2 \phi & 0 & 0 \\ 0 & u^2 + \epsilon^2 \sin^2 \phi & 0 \\ 0 & 0 & \frac{(u^2 + \epsilon^2 \sin^2 \phi)}{(u^2 + \epsilon^2)} \end{bmatrix}$$

$$:= g_{nm} \forall n, m = 1, 2, 3 \quad (\text{A10})$$

3. Laplacean

$$\Delta = \frac{1}{\sqrt{g}} \left\{ \frac{\partial}{\partial \lambda} \left( \frac{\sqrt{g}}{g_{11}} \frac{\partial}{\partial \lambda} \right) + \frac{\partial}{\partial \phi} \left( \frac{\sqrt{g}}{g_{22}} \frac{\partial}{\partial \phi} \right) + \frac{\partial}{\partial u} \left( \frac{\sqrt{g}}{g_{33}} \frac{\partial}{\partial u} \right) \right\}$$

$$= \frac{1}{u^2 + \epsilon^2 \sin^2 \phi} \left\{ \frac{u^2 + \epsilon^2 \sin^2 \phi}{(u^2 + \epsilon^2) \cos^2 \phi} \frac{\partial^2}{\partial \lambda^2} - \tan \phi \frac{\partial}{\partial \phi} \right.$$

$$\left. + \frac{\partial^2}{\partial \phi^2} + 2u \frac{\partial}{\partial u} + (u^2 + \epsilon^2) \frac{\partial^2}{\partial u^2} \right\} \quad (\text{A11})$$

## Appendix B

*Normalised associated Legendre functions of first and second kind*

Here we define the normalised associated Legendre functions of the first kind  $P_{nm}^*(\sin \phi)$  by means of recurrence relations as follows.

$$P_{nm}^*(\sin \phi) = \frac{\sqrt{2n+1}}{\sqrt{2n}} \cos \phi P_{n-1,n-1}^*(\sin \phi) \quad (\text{B1})$$

$$P_{n,n-1}^*(\sin \phi) = \frac{\sqrt{2n+1}}{\sqrt{2(n-1)}} \cos \phi P_{n-1,n-2}^*(\sin \phi) \quad (\text{B2})$$

$$P_{nm}^*(\sin \phi) = \frac{\sqrt{4n^2 - 1}}{\sqrt{n^2 - m^2}} \sin \phi P_{n-1,m}^*(\sin \phi)$$

$$- \frac{\sqrt{(2n+1)(n+m-1)(n-m-1)}}{\sqrt{(n^2 - m^2)(2n-3)}} \cdot P_{n-2,m}^*(\sin \phi) \quad (\text{B3})$$

subject to

$$\forall n \in [3, \infty) \text{ and } m \in [0, n-2]$$

with starting values

$$P_{00}^*(\sin \phi) = 1 \quad (\text{B4})$$

$$P_{10}^*(\sin \phi) = \sqrt{3} \sin \phi \quad (\text{B5})$$

$$P_{11}^*(\sin \phi) = \sqrt{3} \cos \phi \quad (\text{B6})$$

$$P_{20}^*(\sin \phi) = \frac{\sqrt{5}}{2} (3 \sin^2 \phi - 1) \quad (\text{B7})$$

$$P_{21}^*(\sin \phi) = \sqrt{15} \sin \phi \cos \phi \quad (\text{B8})$$

$$P_{22}^*(\sin \phi) = \frac{\sqrt{15}}{2} \cos^2 \phi \quad (\text{B9})$$

The associated Legendre functions of the second kind can be defined by an integral relation of the type

$$Q_{nm}^* \left( \frac{u}{\epsilon} \right) = i^{n+1} Q_{nm} \left( i \frac{u}{\epsilon} \right) \quad (\text{B10})$$

$$Q_{nm} \left( i \frac{u}{\epsilon} \right) = \frac{(-1)^m 2^n (n+m)! m!}{i^{n+1} (n-m)! (2m)!} \left( \frac{u^2 + \epsilon^2}{\epsilon^2} \right)^{m/2}$$

$$\cdot \int_0^\infty \frac{\sinh^{2m} \tau d\tau}{\left( \frac{u}{\epsilon} + \frac{\sqrt{u^2 + \epsilon^2}}{\epsilon} \cosh \tau \right)^{n+m+1}} \quad (\text{B11})$$

However, in practice, instead of the above integral formulas the associated Legendre functions of the second kind are calculated via the recursive relations which enjoy numerical stability, especially for the higher degrees and orders (Thong and Grafarend 1989; Sona 1996).

$$Q_{n|m|}^* \left( \frac{u}{\epsilon} \right) = \sum_{k=0}^{k_{\max}} Q_{n|m|k}^*(u) \quad (\text{B12})$$

$$Q_{n|m|k}^*(u) = \frac{\epsilon^2 (1-n-|m|-2k)(n+|m|+2k)}{2k(2n+2k+1)u^2} \cdot Q_{n|m|k-1}^*(u) \quad \forall k \geq 1 \quad (\text{B13})$$

$$Q_{n|m|0}^*(u) = \cosh^{|m|} \eta \left( \frac{a}{u} \right)^{n+1}$$

$$\forall n \in \mathbb{N}, m \in [-n, n] \subset \mathbb{Z} \quad (\text{B14})$$

The summation of Eq. (B12) is continued until

$$Q_{n|m|k_{\max}}^*(u) - Q_{n|m|k_{\max}-1}^*(u) < \sigma \quad (\text{B15})$$

with starting values for  $n = 0, 1, 2$  and  $m = 0$

$$Q_0^* \left( \frac{u}{\epsilon} \right) = \arccot \left( \frac{u}{\epsilon} \right) \quad (\text{B16})$$

$$Q_1^* \left( \frac{u}{\epsilon} \right) = 1 - \frac{u}{\epsilon} \arccot \left( \frac{u}{\epsilon} \right) \quad (\text{B17})$$

$$Q_2^* \left( \frac{u}{\epsilon} \right) = \frac{1}{2} \left[ \left( 3 \frac{u^2}{\epsilon^2} + 1 \right) \arccot \left( \frac{u}{\epsilon} \right) - 3 \frac{u}{\epsilon} \right] \quad (\text{B18})$$

In Eq. (B15)  $\sigma$  can be selected according to the required accuracy. In our calculations double-precision accuracy, i.e.  $\sigma = 1\text{E} - 16$ , was adopted.

## Appendix C

### *Transformation of spherical harmonic coefficients into spheroidal/ellipsoidal harmonic coefficients*

Nowadays it is common practice to represent the ‘Standard Gravity Earth Models’ in terms of spherical harmonics. Fortunately, precise transformation relations between spherical and ellipsoidal harmonic coefficients are available and therefore we can transfer the spherical harmonic coefficients into ellipsoidal ones without any loss of accuracy. Lemma C1 offers a summary of the transformation of spherical harmonic coefficients into ellipsoidal harmonic coefficients according to Jekeli (1981, 1988). Contributions made by Gleason (1988, 1989); Sona (1996); Yu and Cao (1996) should also be acknowledged.

**Lemma C1** (*Transformation of spherical harmonic coefficients into ellipsoidal harmonic coefficients*). *Spherical harmonic coefficients,  $u_{n,m}$  (spherical), can be uniquely transformed into ellipsoidal harmonic coefficients,  $u_{n,m}$  (ellipsoid) via*

$$u_{n,m}(\text{ellipsoidal}) = Q_{n,|m|}^* \left( \frac{b}{\epsilon} \right) \sum_{l=0}^{(n-m)/2} \lambda_{n,|m|,l} u_{n-2|m|,|m|}(\text{spherical}) \quad (\text{C1})$$

$$\lambda_{n,m,l} = \frac{(2n-2l)!n!}{(2n)!l!(n-1)!} \left[ \frac{(2n-4l+1)(n-m)(n+m)!}{(2n+1)(n-2l+m)!(n-2l-m)!} \right]^{1/2} \cdot \left( \frac{\epsilon}{a} \right)^{2l} \quad (\text{C2})$$

$$\forall n \in [0, \infty) \text{ and } m \in [-n, +n] \quad (\text{C3})$$

By expanding the factorials in Eq. (C2), we can obtain the following recursive formula, which is numerically stable especially for high degrees and orders  $n/m$ .

$$\lambda_{n,m,k} = \frac{((2n-4l+1)(n-2l-m+1)(n-2l-m+2)(n-2l+m+1)(n-2l+m+2))^{1/2}}{2k(2n-2l+1)(2n-4l+5)^{1/2}} \times \left( \frac{\epsilon}{a} \right)^{2l} \lambda_{n,m,l-1} \quad (\text{C4})$$

with the start value

$$\lambda_{n,m,0} = 1 \quad \forall n, m \quad (\text{C5})$$

$Q_{n,|m|}^*(b/\epsilon)$  are associated Legendre functions of the second kind; see Eq. (B12) for the recursive relation.

As can be seen from Eqs. (C1)–(C5), any ellipsoidal harmonic coefficient,  $u_{n,m}$  (spheroid) is equal to the spherical harmonic coefficient of the same degree and order  $u_{n,m}$  (sphere) plus a linear combination of spherical harmonic coefficients of a lower degree but the same order.

For details on transformation of spherical harmonic coefficients into ellipsoidal ones, we refer to Ardalan and Grafarend (2000). The ellipsoidal harmonic coefficients

in different permanent tide systems can be accessed also from <http://www.uni-stuttgart.de/gi/research/index.html#projects>.

## References

- Ardalan AA, Grafarend EW (2000) Reference ellipsoidal gravity potential field and gravity intensity field of degree/order 360/360 (manual of using ellipsoidal harmonic coefficients ‘ellipfree.dat’ and ‘ellipmean.dat’). <http://www.uni-stuttgart.de/gi/research/index.html#projects>
- Burša M, Kouba J, Raděj K, True SA, Vatr V, Vojtišková M (1997a) Monitoring geoidal potential on the basis of TOPEX/POSEIDON altimeter data and EGM96. Paper presented at scientific assembly of IAG, Rio de Janeiro
- Burša M, Raděj K, Šima Z, True SA, Vatr V (1997b) Determination of the geopotential scale factor from TOPEX/POSEIDON satellite altimetry. *Stud Geophys Geod* 41: 203–216
- Burša M, Kouba J, Raděj K, True SA, Vatr V, Vojtišková M (1998) Mean earth’s equipotential surface from TOPEX/POSEIDON altimetry. *Studia Geophys Geod* 42: 459–466
- Burša M, Kouba J, Müller A, Raděj K, True SA, Vatr V, Vojtišková M (1999) Differences between mean sea levels for the pacific, Atlantic and Indian oceans from TOPEX/POSEIDON altimetry. *Studia Geophys Geod* 43: 1–6
- Burša M, Kouba J, Muneendra K, Müller A, Raděj K, True SA, Vatr V, Vojtišková M (2000) Geoidal geopotential and world height system. *Studia Geophys Geod* 43: 327–337
- Eitschberger B, Grafarend EW (1974) World geodetic datum WD 1 and WD 2 from satellite and terrestrial observations. *Bull Géod* 114: 364–385
- Ekman M (1996) The permanent problem of the permanent tide: what to do with it in geodetic reference systems. *Mar Terres* 125: 9508–9513
- Eringen AC (1962) *Nonlinear theory of continuous media*. McGraw-Hill, New York
- Gauss CF (1828) *Bestimmung des Breitenunterschiedes zwischen den Sternwarten von Göttingen und Altona*. Vandenhoeck & Ruprecht, Göttingen
- Gleason DM (1988) Comparing corrections to the transformation between the geopotential’s spherical and ellipsoidal spectrum. *Manuscr Geod* 13: 114–129
- Gleason DM (1989) Some notes on the evaluation of ellipsoidal and spheroidal harmonic expansions. *Manuscr Geod* 14: 114–116
- Grafarend EW, Ardalan AA (1997)  $W_0$ —an estimate in the Finnish Height Datum N60, epoch 1993.4, from twenty-five GPS points of the Baltic Sea Level project. *J Geod* 71: 673–679
- Grafarend EW, Ardalan AA (1999) *World Geodetic Datum 2000*. *J Geod* 73: 611–623
- Groten E (2000) Parameters of common relevance of astronomy, geodesy, and geodynamics. The geodesist’s handbook. *J Geod* 74: 134–140
- Heck B, Rummel R (1990) Strategies for solving the vertical datum problem using terrestrial and satellite geodetic data. In: Sünkel H, Baker T (eds) *Sea surface topography and the geoid*. Springer, Berlin Heidelberg New York, pp 116–128
- Jekeli C (1981) The downward continuation to the Earth’s surface of truncated spherical and ellipsoidal harmonic series of the gravity and height anomalies. Rep 323, Department of Geodetic Science and Surveying, The Ohio State University, Columbus
- Jekeli C (1988) The exact transformation between ellipsoidal and spherical harmonic expansions. *Manuscr Geod* 13: 106–113
- Kakkuri J (1990) Final results of the Baltic Sea level 1990 GPS campaign. *Rep Finnish Geod* 94: 2
- Kakkuri J (1995a) Final results of the Baltic Sea level 1993 GPS campaign. *Rep Finnish Geod* 95: 2
- Kakkuri J (1995b) The Baltic Sea Level project. *Allg Vermess Nachr* 8–9: 331–336

- Kakkuri J (2000) A new best fitting geoid to MSL. Unpublished geoid.
- Kakkuri J, Poutanen M (1997) Geodetic determination of the surface topography of the Baltic Sea. *Mar Geod* 20: 307–316
- Kakkuri J, Kukkamäki TJ, Levallois JJ, Moritz H (1986) Le 250<sup>E</sup> Anniversaire de la Mesure de L'Arc du Meridien en Laponie. Rep 103, Finnish Geodetic Institute, Helsinki
- Lemoine FG, Smith DE, Kunz L, Smith R, Pavlis EC, Pavlis NK, Klosko SM, Chinn DS, Torrence MH, Williamson RG, Cox CM, Rachlin KE, Wang YM, Kenyon SC, Salman R, Trimmer R, Rapp RH, Nerem RS (1996) The development of the NASA GSFC and NIMA joint geopotential model. In: Segawa J, Fujimoto H, Okubo S (eds) *Gravity, geoid and marine geodesy*. Int Assoc Geod Symp vol 117. Springer, Berlin Heidelberg New York 1998, pp 461–469
- Listing, JB (1873) Über unsere jetzige Kenntnis der Gestalt und Größe der Erde. Dietrichsche Verlagsbuchhandlung, Göttingen
- Nesvorný D, Šima Z (1994) Refinement of the geopotential scale factor  $R_0$ , on the satellite altimetry basis. *Earth, Moon, Plan* 65: 79–88
- Poutanen M, Kakkuri J (1999) Final results of the Baltic Sea level 1997 GPS campaign. Rep Finnish Geod Inst 99: 2
- Poutanen M, Malkin Z, Voinov A, Liebsch G, Pan M (1999) Combined solution of the Baltic Sea level 1997 GPS campaign. In: Poutanen M, Kakkuri J (eds) *Final results of the Baltic Sea level 1997 GPS campaign*. Rep Finnish Geod Inst 99: 4, pp 9–40
- Rapp RH (1994) Separation between reference surfaces of selected vertical datums. *Bull Geod* 69: 26–31
- Rummel R, Ilk KH (1995) Height datum connection – the ocean part. *Allg Vermess Nachr* 8–9: 321–330
- Rummel R, Teunissen P (1988) Height datum definition, height datum connection and the role of the geodetic boundary value problem. *Bull Geod* 62: 477–498
- Sansò F, Usai S (1995) Height datum and local geodetic datums in the theory of geodetic boundary value problems. *Allg Vermess Nachr* 8–9: 343–355
- Smith JR (1986) From plane to spheroid. Landmark Enterprises, Rancho Cordova, USA
- Sona G (1996) Numerical problems in the computation of ellipsoidal harmonics. *J Geod* 70: 117–126
- Thong NC, Grafarend EW (1989) A spheroidal model of the terrestrial gravitational field. *Manuscr Geod* 14: 285–304
- Tobé E (1986) Fransysk visit i Tornedalen. 1736–1737, I-Tryck AB, Luleå 1966
- Vermeer M (1995) Two new geoids determined at the FGI. Rep Finnish Geod Inst 95: 5
- Yu J, Cao H (1996) Elliptic harmonic series and original stokes problem with the boundary of the reference ellipsoid. *J Geod* 70: 431–439
- Xu P (1992) A quality investigation of global vertical datum connection. *Geophys J Int* 110: 361–370
- Xu P, Rummel R (1991) A quality investigation of global vertical datum connection. Netherlands Geodetic Commission, New Series, 34, Delft